Assignment II

1. Verify that $p(x) = x^2-1$ and $q(x) = x^3+3x^2-x-3$ interpolate the data $(-2,3), (-1,0), (1,0)$. Explain why this does not violate the uniqueness part of the theorem on existence of polynomial interpolation.

2. Express the polynomial $p(x) = 3-(x-1)\Big(4-(x+1)\big(5-x(6-(x+2))\big)\Big)$ in Newton's form with centres $2, 2, 1, -1$ and hence find its derivative at $x = 2$. Also, find its second derivative at $x = 2$.

3. Given that $\omega_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$. Show that
$$f[x_0, x_1, \cdots, x_n] = \sum_{k=0}^{n} \frac{f(x_k)}{\omega'_{n+1}(x_k)}.$$

4. Given that $\omega_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$, where $x_i = x_0 + ih$, $i = 0, 1, 2, \cdots, n$ and $h = (x_n - x_0)/n$. If $x \in [x_0, x_n]$, then show that $|\omega_{n+1}(x)| \leq \frac{1}{4}h^{n+1}n!$. You may use that fact that $(n-j)!(j+1)! \leq n!$ for $0 \leq j \leq n-1$.

5. If $p(x) = x^4 + x^3 + x^2 + x + 1$ is written as
$p(x) = c_0 + c_1 x + c_2 x(x+1) + c_3 x(x+1)(x-1) + c_4 x(x+1)(x-1)(x-2)$,
then find the value of $c_0 + c_1 + c_2 + c_3 + c_4$

6. The Lagrange interpolation polynomial $p_n(x)$ for $f(x)$ at x_i, for $i = 0, 1, \cdots, n$ is given by $p_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$. Prove that L_i's satisfy $\sum_{i=0}^{n} L_i(x) = 1$.

7. If $(y_0, y_1, \cdots, y_n)$ is a permutation of $(x_0, x_1, \cdots, x_n)$, then show that $f[y_0, y_1, \cdots, y_n] = f[x_0, x_1, \cdots, x_n]$.

8. Let $g(x) = f[x_0, x_1, \cdots, x_n, x]$. Prove that
$$g[y_0, y_1, \cdots, y_m] = f[x_0, x_1, \cdots, x_n, y_0, y_1, \cdots, y_m]$$

9. It can be proved that $f[x_0, x_1, \cdots, x_k]$ is a continuous function of its arguments. If $g(x) = f[x_0, x_1, \cdots, x_m, x]$, then prove that
$$g^{(n)}(x) = n! f[x_0, x_1, x_2, \cdots, x_m, x, \cdots, x],$$
where the argument x is repeated $n+1$ times.

10. Complete the following divided difference table and use them to find the polynomial of degree ≤ 3 that interpolates the function values given.

x	f[]	f[,]	f[, ,]	f[, , ,]
-1	2			
1	f_1		2	
x_2	6			
		2		
5	10			

11. A polynomial $p(x) = x^3 - x + 2$ has the following values:

x	-1	0	1	2	3
p(x)	2	2	2	8	26

Find the polynomial $q(x)$ which interpolates the following data

x	-1	0	1	2	3
q(x)	2	2	2	8	-46

12. Given that $f(1) = 1, f(2) = 3$, and $f(4) = 3$, use Lagrange's interpolation formula to estimate the value of $f(3)$.

13. Use Newton's divided difference formula to show that it is quite invalid to interpolate $\sqrt[3]{20}$ from the points $(0,0), (1,1), (8,2), (27,3), (64,4)$ of $f(x) = \sqrt[3]{x}$

14. Given that $e^0 = 1, e^{0.5} = 1.64872$, and $e^1 = 2.71828$, use Newton's divided difference formula to estimate the value of $e^{0.25}$. Find lower and upper bounds on the magnitude of the error and verify that the actual magnitude is within the calculated bounds.

15. Construct the Lagrange interpolation polynomial p_1 of degree 1, for a continuous function f defined on the interval $[-1,1]$, using the interpolation points $x_0 = -1, x_0 = 1$. Show further that if the second derivative of f exists and is continuous on $[0,1]$, then

$$|f(x) - p_1(x)| \leq \frac{M_2}{2}(1 - x^2) \leq \frac{M_2}{2}, \quad x \in [-1,1]$$

where $M_2 = \max_{x\in[-1,1]} |f''(x)|$. Give an example of a function f, and a point x, for which equality is achieved.

16. The interpolation error for a sufficient smooth function $f(x)$ is given by

$$f(x) - p_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!}\omega_{n+1}(x), \quad \xi \in (a,b).$$

   (a) Write down the Lagrange interpolation polynomial of degree 1 for the function $f(x) = x^3$, using the points $x_0 = 0, x_1 = a$. By direct calculation, verify the above error formula and show that ξ is unique and has the value $\xi = (x+a)/3$.

   (b) Repeat the calculation for the function $f(x) = (2x-a)^4$; show that in this case there are two possible values for ξ, and give their values.

17. Consider a table of natural logarithm values, for the interval $[1/2, 1]$. How many entries do we have to have in the table for linear interpolation between entries to be accurate to within 10^{-3}?

18. The function defined by

$$f(x) = \int_0^x \sin s^2 \, ds$$

has been tabulated for equally spaced values of x with step $h = 0.1$. What is the maximum error encountered if cubic interpolation is to be used to calculate $f(\bar{x})$ for $\bar{x}$ at any points on the interval $[0, \pi/2]$?

19. Consider $n+1$ interpolation points $x_0, x_1, \cdots, x_n$. An interpolating spline of degree n is required to have continuous derivatives up to and including $n-1$ at the knots. How many additional conditions are required to specify the spline uniquely.

20. For what values of k is the following a spline function?

$$q(x) = \begin{cases} x^3 - x^2 + kx + 1, & 0 \le x \le 1 \\ -x^3 + (k+2)x^2 - kx + 3, & 1 \le x \le 2 \end{cases}$$

21. Is the following function $q(x)$ a natural cubic spline on the interval $-1 \le x \le 2$?

$$q(x) = \begin{cases} 2(x+1) + (x+1)^3, & x \in [-1, 0] \\ 3 + 5x + 3x^2, & x \in [0, 1] \\ 11 + 11(x-1) + 3(x-1)^2 - (x-1)^3, & x \in [1, 2] \end{cases}$$

22. Determine whether the following is a spline function:

$$q(x) = \begin{cases} x, & x \in [-3, 1] \\ -\frac{1}{2}(2-x)^2 + 3/2, & x \in [1, 2] \\ \frac{3}{2}, & x \in [2, 5]. \end{cases}$$

23. Determine a, b, c so that the following function

$$q(x) = \begin{cases} x^3, & x \in [0, 1] \\ -\frac{1}{2}(x-1)^3 + a(x-1)^2 + b(x-1) + c, & x \in [1, 3] \end{cases}$$

is a cubic spline. Is it a natural cubic spline?

24. Find the natural cubic spline that interpolates the data $(1, 1)$, $(2, 1)$ and $(3, 0)$.

25. Give an example of a cubic spline with knots $0, 1, 2$, and 3 that is linear in $[0, 1]$, cubic in $[1, 2]$, and quadratic in $[2, 3]$.

26. Determine a, b, c, d so that the following function

$$q(x) = \begin{cases} x^3 + 1, & x \in [-5, 0] \\ ax^3 + bx^2 + cx + d, & x \in [0, 5] \end{cases}$$

is a cubic spline which assumes the value 5 at $x = 1$.