Assignment-III

1. Prove that when the matrix A is nonsingular, then

$$\frac{1}{||A^{-1}||} = \min_{||x||=1} ||Ax||$$

2. Prove that if $||A|| < 1$ then $I - A$ is invertible and

$$(I-A)^{-1} = I + A(I-A)^{-1}$$

Hence deduce that

$$||(I-A)^{-1}|| \leq \frac{1}{1-||A||}$$

Also show that if A is invertible and $||A-B|| < 1/||A^{-1}||$, then B is also invertible.

3. Show that for any matrix $A_{m\times n}$,

$$||A||_\infty \leq \sqrt{n}||A||_2 \quad \text{and} \quad ||A||_2 \leq \sqrt{m}||A||_\infty$$

4. Suppose that $A_{n\times n}$ is a nonsingular matrix and $b \in R^n_* = R^n\backslash\{0\}$. Given that $Ax = b$ and $(A+\delta A)(x+\delta x) = b + \delta b$, and that $||A^{-1}||\,||\delta A|| < 1$. Show that

$$\frac{||\delta x||}{||x||} \leq \frac{\text{cond}(A)}{1-\text{cond}(A)\frac{||\delta A||}{||A||}}\left(\frac{||\delta b||}{||b||} + \frac{||\delta A||}{||A||}\right)$$

5. Consider the following tridiagonal matrix

$$A = \begin{pmatrix} b_1 & c_1 & \cdots & \cdots & \cdots \\ a_2 & b_2 & c_2 & \cdots & \cdots \\ \cdots & a_3 & b_3 & c_3 & \cdots \\ \vdots & & & & \\ \cdots & \cdots & \cdots & a_n & b_n \end{pmatrix}$$

with $|b_1| > |c_1|$, $|b_n| \geq |a_n|$ and $|b_j| \geq |a_j| + |c_j|$, $j = 2, 3, \cdots, n-1$. If $a_j \neq 0$ for $j = 2, 3, \cdots, n$, then show that every principal submatrix (and hence A) is nonsingular.

6. Calculate the number of multiplications and divisions required to solve $Ax = b$ by LU factorization where A is the tridiagonal matrix given above.

7. A matrix A is (strictly) diagonally dominant if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \qquad i = 1, 2, \cdots, n$$

(i) Show that Gauss elimination without partial pivoting preserve the diagonal dominance of a matrix.

(ii) Show that if Gauss elimination is carried out with scaled partial pivoting on a diagonally dominant matrix, then row interchange is not necessary.

(Thus, if a matrix is diagonally dominant then it has LU factorization.)

8. A 3×3 matrix A has LU decomposition $A = LU$ where

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -9 & -15 \\ 0 & 0 & -6 \end{bmatrix}$$

If $Ly = b$ where $b = [1, 2, -1]^T$ has the solution $y = [1, -5, -5]^T$, the find the solution to $Ax = b$.

9. The spectral radius of a matrix A is denoted by $\rho(A)$ and is defined by

$$\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$$

It can be shown that $||A||_2 = \sqrt{\rho(A^T A)}$. The condition number of a matrix A is defined by $\kappa(A) = ||A||\ ||A^{-1}||$. There is a condition number for each norm; for example, if we use the 2−norm, then $\kappa_2(A) = ||A||_2 ||A^{-1}||_2$.

(a) Prove that for any nonsingular matrix A

$$\kappa_2(A) = \left(\frac{\lambda_n}{\lambda_1}\right)^{1/2}$$

where λ_1 is the smallest and λ_n is the largest eigenvalue of the matrix $A^T A$

(b) Show that the condition number $\kappa_2(A)$ of an orthogonal matrix is equal to 1.

(c) Show that if λ is an eigenvalue of $A^T A$, then

$$0 \le \lambda \le ||A^T||\ ||A||,$$

provided that the same norm is used for both A and A^T. Hence show that, for any nonsingular matrix A

$$\kappa_2(A) \le [\kappa_1(A)\kappa_\infty(A)]^{1/2}$$

10. Prove that if A is diagonally dominant, then both the Jacobi and Gauss-Seidel method converges.

11. For solving $Ax = b$ we use $x^{(k+1)} = Gx^{(k)} + c$ where G is the iteration matrix. Consider the matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Find the iteration matrix and its spectral radius for the Jacobi and Gauss-Seidel schemes. Also, find the iteration matrix for the SOR scheme

12. Find the iteration matrix for both the Jacobi and Gauss-Seidel method in solving $Ax = b$, where the tridiagonal matrix A is given in Q.5 with $a_i = c_i = -1$ and $b_i = 2$ for all i.

13. Show that for the SOR method to converge, the relaxation parameter ω must satisfy $0 < \omega < 2$.

14. Consider the linear system $Ax = b$ where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Show that A is positive definite. Starting with $x^{(0)} = (0, 0, 0)^T$, show that CG method converges in 3 steps.