Hermite Interpolation

Hermite interpolation interpolates function values and function derivatives at the interpolation points. Let the interpolation points be $x_i, i = 0, 1, 2, \cdots, n$. Let the Hermite interpolation polynomial be

$$p_{2n+1}(x) = \sum_{i=0}^{n} \Big(H_i(x) f(x_i) + K_i(x) f'(x_i) \Big), \tag{1}$$

where both $H_i(x)$ and $K_i(x)$ are polynomial of degree $2n+1$. Further, to satisfy the interpolation conditions, we need

$$H_i(x_j) = \delta_{ij}, \quad H_i'(x_j) = 0,$$

and

$$K_i(x_j) = 0, \quad K_i'(x_j) = \delta_{ij}.$$

Now

$$l_i(x) = \frac{\omega_{n+1}(x)}{(x - x_i)\omega_{n+1}'(x_i)}$$

is a polynomial of degree n and $l_i(x_j) = \delta_{ij}$. Let us choose $H_i(x) = r_i(x) l_i^2(x)$ and $K_i(x) = s_i(x) l_i^2(x)$. Then to satisfy the conditions, we need to impose

$$r_i(x_i) = 1, \quad r_i'(x_i) + 2l_i'(x_i) = 0,$$

and

$$s_i(x_i) = 0, \quad s_i'(x_i) = 1.$$

Hence, $r_i(x) = 1 - 2l_i'(x_i)(x - x_i)$ and $s_i(x) = x - x_i$. Hence,

$$H_i(x) = \Big(1 - 2l_i'(x_i)(x - x_i)\Big) l_i^2(x), \quad K_i(x) = (x - x_i) l_i^2(x).$$

To prove that this polynomial is the unique polynomial of degree less than or equal to $2n+1$, let there exists another polynomial $q_{2n+1}(x)$. Then $d(x) = p_{2n+1}(x) - q_{2n+1}(x)$ is a polynomial of degree less than or equal to $2n+1$. Since $d(x_i) = 0$ for $i = 0, 1, 2, \cdots, n$, it follows from Rolle's theorem that $d'(x)$ has n zeros that lie in the intervals (x_{i-1}, x_i) for $i = 1, 2, \cdots, n$. Further, since $d'(x_i) = 0$, it is clear that $d'(x)$ has additional $n+1$ zeros. Hence, $d'(x)$ has $2n+1$ distinct zeros. But $d'(x)$ is a polynomial of degree less than or equal to $2n$. Hence $d(x) \equiv 0$.

To find the error formula, let $\bar{x}$ be a point different from x_i's and let

$$f(\bar{x}) - p_{2n+1}(\bar{x}) = c(\bar{x}) \omega_{n+1}^2(\bar{x}).$$

Now consider the function

$$\phi(t) = f(t) - p_{2n+1}(t) - c(\bar{x}) \omega_{n+1}^2(t)$$

Then $\phi(x_i) = 0$ for $i = 0, 1, 2, \cdots, n$ and $\phi(\bar{x}) = 0$. Let $a = \min_{0 \le i \le n}\{\bar{x}, x_i\}$ and $b = \max_{0 \le i \le n}\{\bar{x}, x_i\}$. Now by Roll's theorem, $\phi'(t)$ vanishes $(n+1)$ times in the interior of the intervals formed by x_i's and $\bar{x}$. Further $\phi'(t)$ vanishes at the $(n+1)$ points x_i. Clearly, $\phi'(t)$ has $(2n+2)$ distinct zeros in $[a, b]$. By Rolle's theorem $\phi^{(2n+2)}(\xi) =$ for $\xi \in (a, b)$. This implies

$$f^{(2n+2)}(\xi) - c(\bar{x})(2n+2)! = 0.$$

Hence

$$f(\bar{x}) - p_{2n+1}(\bar{x}) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \omega_{n+1}^2(\bar{x}).$$