Lab Assignment IV

LU decomposition is one of the efficient way to solve a system of linear equations $Ax = b$ for many different choice of b. We express A as $A = LU$ where L and U are lower and upper triangular matrices respectively. Then $Ax = b$ is equivalent to $LUx = b$ which can be solved by $Lz = b$ followed by $Ux = z$. Note that $Lz = b/Ux = z$ can be solved by forward/backward substitution. We shall apply this method to solve system of linear equations $Ax = d$ where A is a tridiagonal matrix of the form

$$A = \begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{bmatrix}$$

Note that we can define the tridiagonal matrix using three vectors a, b, c each of dimension n where $a_1 = c_n = 0$. We take L and U as

$$L = \begin{bmatrix} 1 & & & & & \\ \alpha_2 & 1 & & & & \\ & \alpha_3 & 1 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & & & 1 & \\ & & & & \alpha_n & 1 \end{bmatrix} \qquad U = \begin{bmatrix} \beta_1 & c_1 & & & & \\ & \beta_2 & c_2 & & & \\ & & \beta_3 & c_3 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & & & \beta_{n-1} & c_{n-1} \\ & & & & & \beta_n \end{bmatrix}.$$

Clearly each of L and U can be stored using a one dimensional array. The unknowns α_i and β_i satisfy $\beta_1 = b_1$ and $\alpha_j \beta_{j-1} = a_j$, $\beta_j + \alpha_j c_{j-1} = b_j$ for $j = 2, \cdots, n$. Solution of $Lz = d$ is obtained from $z_1 = d_1$ and $\alpha_j z_{j-1} + z_j = d_j$ for $j = 2, \cdots, n$. Finally $Ux = z$ is solved as $x_n = z_n/\beta_n$ and $\beta_j x_j + c_j x_{j+1} = z_j$ for $j = n-1, n-2, \cdots, 1$. (Note that in C, the array index starts from 0). To check the accuracy, you may assign 1 to all a's and c's and 2 to all b's. Further, assign 4 to all the d's except $d_1 = d_n = 3$. Then all the x's of the solution are 1.

Implement the above algorithm for the system

$$\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & & & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix},$$

where $d_i = 6ih^3$ for $i = 1, \cdots, n-1$ and $d_n = 6nh^3 - 2$ with $h = 1/(n+1)$.

Below is a typical input/output:

```
Enter n:9
--------------
i     x[i]
--------------
1  0.101000
2  0.208000
3  0.327000
4  0.464000
5  0.625000
6  0.816000
7  1.043000
8  1.312000
9  1.629000
```