MTH308: QUIZ-1A

1. Find $a, b, c, \alpha, \beta, \gamma$ so that

$$S(x) = \begin{cases} ax^3 + bx^2 + cx + 2, & x \in [-1, 0] \\ x^3 + \alpha x^2 + \beta x + \gamma, & x \in [0, 1]. \end{cases}$$

is a natural cubic spline for $(-1, -5), (0, 2)$ and $(1, 5)$. [7]

Ans. We have $S''(1) = 0 \implies \alpha = -3$, $S(0) = 2 \implies \gamma = 2$, $S(1) = 5 \implies \beta = 5$. Also, $S'(0+) = S'(0-) \implies c = \beta = 5$ and $S''(0+) = S''(0-) \implies b = \alpha = -3$. Finally, $S''(-1) = 0 \implies -6a + 2b = 0 \implies a = -1$.
Thus, $a = -1, b = -3, c = 5$ and $\alpha = -3, \beta = 5, \gamma = 2$.
(All correct 7 marks. Otherwise 1 mark for one parameter)

2. For an n-vector x and an $n \times n$ matrix A, let $||A||$ be the matrix norm induced by the vector norm $||x||$. Show that $||Ax|| \leq ||A|| ||x||$ for all x. Further, if A is nonsingular and B is any $n \times n$ singular matrix, then show that $||A^{-1}|| \, ||A - B|| \geq 1$. [6]

Ans. From definition, $||A|| = \max_{x \neq 0} ||Ax||/||x||$. Hence, $||Ax||/||x|| \leq ||A||$ for $x \neq 0 \implies ||Ax|| \leq ||A|| \, ||x||$. For $x = 0$, equality holds.
Since B is singular, $\exists\, x \neq 0$ such that $Bx = 0$ and hence $Ax = (A - B)x$
$\implies x = A^{-1}(A - B)x \implies ||x|| \leq ||A^{-1}|| \, ||A - B|| \, ||x||$. Now $||x|| > 0$ and hence the result follow. (Three marks for each part)

3. For the function $f(x) = \ln(1 + x) - \ln(x)$, show that loss-of-significance error occurs in the evaluation for certain value of x. Without using Taylor series, reformulate the evaluation to minimize the error? [3]

Ans. $1 + x \approx x$ for large values of x and hence loss-of-significance error occurs at large value of x. To reduce, we use $\log(1 + 1/x)$ for large x.
(one mark for first part and two marks for second part)

4. Consider the matrix [4]

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{4} \end{pmatrix}$$

Does the solution of $Ax = b$ by Jacobi method for any initial $x^{(0)}$ converge? Justify your answer.

Ans. We have

$$\begin{aligned} x_1^{(k+1)} &= (2/3)x_2^{(k)} + 2b_1 \\ x_2^{(k+1)} &= (4/3)x_1^{(k)} + 4b_2 \end{aligned}$$

Thus $x^{(k+1)} = Gx^{(k)} + c$ where

$$G = \begin{pmatrix} 0 & \frac{2}{3} \\ \frac{4}{3} & 0 \end{pmatrix}$$

Eigenvalues of G are $\pm 2\sqrt{2}/3$ and hence $\rho(G) < 1$ and thus iteration converges.
(Two marks for G and two marks for $\rho(G)$. Note that diagonal dominance is not necessary for convergence)