

Dimensional Analysis II

Proof of Pi theorem: We demonstrate two propositions for the equation

$$f(q_1, q_2, \dots, q_n) = 0.$$

A Among the dimensional quantities $q_1, q_2, \dots, q_n$, we can form $n - r$ independent dimensionless variables, where r is the rank of the dimension matrix A .

B Suppose $\Pi_1, \Pi_2, \dots, \Pi_{n-r}$ are $n - r$ dimensionless variables, then the above equation is equivalent to

$$h(\Pi_1, \Pi_2, \dots, \Pi_{n-r}) = 0 \quad \text{for some unknown function } h.$$

Note that we have omitted dimensionless variables that may appear in the function f , which will also appear in the function h .

Proof of proposition A: We denote the dimensions of q_i by square brackets notation $[q_i]$. Let $L_1, L_2, \dots, L_m$ be fundamental dimensions and

$$q_i = L_1^{a_{1i}} L_2^{a_{2i}} \dots L_m^{a_{mi}}$$

Let Π be a quantity constructed as follows:

$$\Pi = q_1^{p_1} q_2^{p_2} \dots q_n^{p_n},$$

for some exponents $p_1, p_2, \dots, p_n$. We want the p_i such that $[\Pi] = 1$, i.e., Π is dimensionless.

$$\begin{aligned} [\Pi] &= [q_1]^{p_1} [q_2]^{p_2} \dots [q_n]^{p_n} \\ &= (L_1^{a_{11}} L_2^{a_{21}} \dots L_m^{a_{m1}})^{p_1} (L_1^{a_{12}} L_2^{a_{22}} \dots L_m^{a_{m2}})^{p_2} \dots (L_1^{a_{1n}} L_2^{a_{2n}} \dots L_m^{a_{mn}})^{p_n} = 1 \end{aligned}$$

This implies

$$\begin{aligned} a_{11}p_1 + a_{12}p_2 + \dots + a_{1n}p_n &= 0 \\ a_{21}p_1 + a_{22}p_2 + \dots + a_{2n}p_n &= 0 \\ &\vdots \\ a_{m1}p_1 + a_{m2}p_2 + \dots + a_{mn}p_n &= 0 \end{aligned}$$

Thus, we obtain a homogeneous system of m equations in n unknowns $p_1, p_2, \dots, p_n$, given in a matrix form by

$$\mathcal{A}p = 0$$

where $\mathbf{p} = [p_1, p_2, \dots, p_n]^t$ is a column vector of the unknown exponents, and the $m \times n$ matrix $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The matrix $\mathcal{A}$ is called the dimension matrix. The elements of the i -th column correspond to the exponents of q_i in terms of the powers of $L_1, L_2, \dots, L_m$. Also, the elements of i -th row correspond to the exponents of L_i in $q_1, q_2, \dots, q_m$. Hence, the number of rows is the number of fundamental dimensions, and the number of columns is the number of dimensional variables. From linear algebra, we know that if r is the rank of the matrix, then the number of independent solutions is $n - r$. We also know that the rank of a matrix is the number of linearly independent rows, which is the number of nonzero rows when the matrix is reduced to row echelon form. Hence, the number of independent dimensionless variables that can be formed from $q_1, q_2, \dots, q_n$, is $n - r$.

Proof of proposition B: This part of the proof is not general. We show the proof for $n = 4, m = 2$, and $r = 2$. Therefore, we have a unit-free physical law of the form

$$f(q_1, q_2, q_3, q_4) = 0,$$

with $q_i = L_1^{a_{1i}} L_2^{a_{2i}}$, $i = 1, 2, 3, 4$.

Thus, we have the dimension matrix

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix},$$

which is of rank $r = 2$. Hence, it has two independent columns. Since it is possible to rearrange the columns corresponding to the dimensional variables q_i , without any loss of generality, we assume that the first two columns are independent. If we construct a dimensionless quantity

$$\Pi = q_1^{p_1} q_2^{p_2} q_3^{p_3} q_4^{p_4},$$

then p_1, p_2, p_3, p_4 satisfy

$$\mathcal{A} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \mathbf{0} \implies p_1 \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + p_2 \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} + p_3 \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} + p_4 \begin{pmatrix} a_{14} \\ a_{24} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since the first two columns of $\mathcal{A}$ are a basis of the column space, each of the last two columns can be written as a linear combination of the first two columns. Thus,

$$\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} = c_{31} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + c_{32} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}, \quad \begin{pmatrix} a_{14} \\ a_{24} \end{pmatrix} = c_{41} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + c_{42} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

Substituting these in the above relation among p_i , we find

$$\implies (p_1 + c_{31}p_3 + c_{41}p_4) \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + (p_2 + c_{32}p_3 + c_{42}p_4) \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since the first two columns are independent, we get

$$p_1 + c_{31}p_3 + c_{41}p_4 = 0, \quad p_2 + c_{32}p_3 + c_{42}p_4 = 0 \implies \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = p_3 \begin{pmatrix} -c_{31} \\ -c_{32} \\ 1 \\ 0 \end{pmatrix} + p_4 \begin{pmatrix} -c_{41} \\ -c_{42} \\ 0 \\ 1 \end{pmatrix}$$

Hence, we get two dimensionless quantities

$$\Pi_1 = q_1^{-c_{31}} q_2^{-c_{32}} q_3, \quad \Pi_2 = q_1^{-c_{41}} q_2^{-c_{42}} q_4 \implies q_3 = \Pi_1 q_1^{c_{31}} q_2^{c_{32}}, \quad q_4 = \Pi_2 q_1^{c_{41}} q_2^{c_{42}}$$

Hence,

$$f(q_1, q_2, q_3, q_4) = f(q_1, q_2, \Pi_1 q_1^{c_{31}} q_2^{c_{32}}, \Pi_2 q_1^{c_{41}} q_2^{c_{42}}) \equiv G(q_1, q_2, \Pi_1, \Pi_2)$$

Thus, the physical law $f(q_1, q_2, q_3, q_4) = 0$ if and only if $G(q_1, q_2, \Pi_1, \Pi_2) = 0$ holds. Since $f(q_1, q_2, q_3, q_4) = 0$ is unit-free, it follows that $G(q_1, q_2, \Pi_1, \Pi_2) = 0$ is also unit-free.

If we use bar to denote the variables in new units, then $\bar{\Pi}_1 = \Pi_1$ and $\bar{\Pi}_2 = \Pi_2$ since the value of a dimensionless quantity remains same under any change of units. Since $G(q_1, q_2, \Pi_1, \Pi_2) = 0$ is unit free, we must have

$$G(\bar{q}_1, \bar{q}_2, \Pi_1, \Pi_2) = 0$$

Since $[q_1] = L_1^{a_{11}} L_2^{a_{21}}$ and $[q_2] = L_1^{a_{12}} L_2^{a_{22}}$, we have

$$\bar{q}_1 = \lambda_1^{a_{11}} \lambda_2^{a_{21}} q_1, \quad \bar{q}_2 = \lambda_1^{a_{12}} \lambda_2^{a_{22}} q_2,$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ are the scale factors for L_1 and L_2 . We claim that we can choose scale factors such that $\bar{q}_1 = 1$ and $\bar{q}_2 = 1$, that imply $\lambda_1^{a_{11}} \lambda_2^{a_{21}} q_1 = 1$ and $\lambda_1^{a_{12}} \lambda_2^{a_{22}} q_2 = 1$. Taking log, we find

$$a_{11} \log \lambda_1 + a_{21} \log \lambda_2 = -\log q_1$$

$$a_{12} \log \lambda_1 + a_{22} \log \lambda_2 = -\log q_2$$

We can uniquely solve for $\log \lambda_1$ and $\log \lambda_2$ since the coefficient matrix (the first two columns of the dimension matrix) is non-singular. Hence, there exists unique λ_1 and λ_2 such that

$$\bar{q}_1 = \lambda_1^{a_{11}} \lambda_2^{a_{21}} q_1 = 1, \quad \bar{q}_2 = \lambda_1^{a_{12}} \lambda_2^{a_{22}} q_2 = 1.$$

Hence, $G(\bar{q}_1, \bar{q}_2, \Pi_1, \Pi_2) = 0$ is equivalent to $h(\Pi_1, \Pi_2) \equiv G(1, 1, \Pi_1, \Pi_2) = 0$.

Example: Suppose a van is being driven on a highway with gusty wind. How does the speed of the van affect the resistance experienced by the van? Suppose the resistance depends on the

vehicle's speed v and the area A of the van exposed to the wind. Assuming $f(R, A, v) = 0$, we construct

$$\Pi = v^a A^b R^c$$

The dimension matrix w.r.t. fundamental dimensions M, L, T is

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 0 & -2 \end{pmatrix}$$

Since the matrix is nonsingular, the only solution is $a = b = c = 0$, i.e., it is not possible to form a dimensionless variable with the given dimensional variables. Let us assume that air density ρ also affects resistance. Assuming $f(R, A, v, \rho) = 0$, we construct

$$\Pi = v^a A^b \rho^c R^d$$

The dimension matrix w.r.t. fundamental dimensions M, L, T is

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 2 & -3 & 1 \\ -1 & 0 & 0 & -2 \end{pmatrix}$$

Clearly, rank $r = 3$, and hence the number of dimensionless variables is $n - r = 1$. Solution of

$$c + d = 0, a + 2b - 3c + d = 0, -a - 2d = 0 \implies \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = d \begin{pmatrix} -2 \\ -1 \\ -1 \\ 1 \end{pmatrix} \implies \Pi = \frac{R}{\rho v^2 A}$$

Hence, the Pi theorem implies that

$$h(\Pi) = 0 \implies \frac{R}{\rho v^2 A} = k \implies R = k \rho A v^2, \quad k \text{ is an unknown constant}$$

This implies that if we increase the speed to twice the original, then the wind resistance will be four times the original.

Example: Consider a long, narrow tube along the direction x . A mass M_0 of diffusive material (e.g., pollutant) is released at $x = 0$ at $t = 0$. Assume that the concentration u is uniform in the y and z directions. We need to find the concentration $u(x, t)$ for $t > 0$. The unit of concentration is mass/length. If D is the diffusion coefficient, then the governing equations are

$$\begin{aligned} \frac{\partial u}{\partial t} &= D \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, t > 0 \\ u(x, 0) &= M_0 \delta(x), & -\infty < x < \infty, \end{aligned}$$

where the Dirac delta function $\delta(x)$ satisfies $\int_I \delta(x) dx = 1$ for any interval I containing $x = 0$. The property $\int_I \delta(x) dx = 1$ implies that $\delta(x)$ has the inverse unit of x , i.e., $\delta(x)$ has the inverse

of length as a unit here. In general, the Dirac delta function has the inverse unit (or dimension) of its argument.

We expect the solution u to depend on x, t, D and M_0 . Hence, there is a relation of the form $f(u, x, t, D, M_0) = 0$. Consider

$$\Pi = x^a t^b M_0^c D^d u^e.$$

The dimension matrix $\mathcal{A}$ with respect to fundamental dimensions M, L and T is

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

Here, rank $r = 3$ and we find

$$c + e = 0, a + 2d - e = 0, b - d = 0 \implies a = -2d + e, b = d, c = -e$$

Hence,

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = d \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + e \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

We construct

$$\Pi_1 = \frac{\sqrt{Dt}}{x}, \quad \Pi_2 = \frac{xu}{M_0}$$

For ease of calculation (less calculation involving partial derivatives w.r.t. x), we take

$$\frac{1}{\Pi_1} = \frac{x}{\sqrt{Dt}} \quad \text{and} \quad \Pi_2 \Pi_1 = \frac{u\sqrt{Dt}}{M_0}$$

as dimensionless variables. Using Pi theorem, we find

$$\frac{u\sqrt{Dt}}{M_0} = g\left(\frac{x}{\sqrt{Dt}}\right) \implies u = \frac{M_0}{\sqrt{Dt}}g(s), \quad s = \frac{x}{\sqrt{Dt}}$$

Thus, the Pi theorem helps us to find the form of the solution $u(x, t)$. Substituting into the governing equation, we find

$$g'' + \frac{1}{2}(sg' + g) = 0 \implies g' + \frac{s}{2}g = C$$

For a fixed $t > 0$, we expect $u(x, t)$ to be an even function of x , and hence $g(s)$ must be an even function. Consequently, $g'(0) = 0$ at $s = 0$. Using this in the above equation, we find $C = 0$.

Thus,

$$\frac{dg}{ds} = -\frac{s}{2}g \implies g(s) = Ae^{-s^2/4}$$

Hence,

$$u(x, t) = \frac{AM_0}{\sqrt{Dt}}e^{-x^2/4Dt}$$

Now, the total mass should be conserved at all time. Thus, for any $t > 0$

$$\int_{-\infty}^{\infty} u(x, t) dx = \int_{-\infty}^{\infty} u(x, 0) dx = M_0,$$

which implies

$$\frac{AM_0}{\sqrt{Dt}} \int_{-\infty}^{\infty} e^{-x^2/4Dt} dx = M_0$$

Substituting $x/2\sqrt{Dt} = w$, we find

$$4A \int_0^{\infty} e^{-w^2} dw = 1 \implies 4A \frac{\sqrt{\pi}}{2} = 1 \implies A = \frac{1}{2\sqrt{\pi}}$$

Finally, the solution to the above problem is

$$u(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}.$$