Stochastic models: Gambler's ruin

Stochastic process: A stochastic process is a family of random variables that represents the evolution of a system over time or space. Stochastic processes are mainly classified based on two features: the time domain (index set) and the state space.

- Discrete-Time, Discrete-State: Here, the index set is discrete, for example, time steps $n = 0, 1, 2, \cdots$ and the state space is also discrete. A Markov chain is an example in this category. The gambler's ruin problem (discussed below) is another example.
- Discrete-Time, Continuous-State: Here, the index set is discrete, and the state space is continuous. The daily closing stock price of a listed share is an example of this.
- Continuous-Time, Discrete-State: Here, the index set is continuous, and the state space is discrete. A Poisson process (e.g., counting telephone call arrivals) is an example.
- Continuous-Time, Continuous-State: Here, both the index set and the state space are continuous. Brownian Motion is an example.

Gambler's ruin: This is an example of a discrete-time and discrete-space stochastic model. Suppose a gambler starts with n rupees and makes a sequence of bets. For each bet, the gambler wins 1 rupee with probability p, loses 1 rupee with probability q, and gets back the original amount with probability r. Obviously, $p + q + r = 1$. (This is similar to the random walk model, in which a walker, at every unit of time, makes a step of exactly one unit to the right with probability p, to the left with probability q, and stays at the same place with probability $r = 1 - p - q$.) The betting is fair if $p = q$, sub-fair if $p < q$, and super-fair if $p > q$. Consider the following rules and facts:

i. The gambler starts with a fortune of n rupees.

ii. The gambler bets 1 rupee in every bet.

iii. The gambler decides to play until he loses it all (fortune becomes zero) or his fortune reaches N rupees (where $N > n$). Thus, the gambler wins $N - n$ in the latter case.

iv. The betting at any step does not depend on the previous history.

Let R be the event that the gambler loses it all before reaching N, and A_k be the random variable that denotes the amount of rupees with the gambler at time step k. Let

$$\pi(n) = \text{prob}(R|A_0 = n)$$

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denote the probability that the gambler goes broke before reaching N, given that he starts with n rupees. Obviously,

$$\pi(0) = \text{prob}(R|A_0 = 0) = 1, \quad \pi(N) = \text{prob}(R|A_0 = N) = 0,$$

which are the boundary conditions. Let W, L, and D be the events that the first bet results in a win, a loss, or a tie (i.e., get-back of the original). Now, we write

$$R = (R \cap W) \cup (R \cap L) \cup (R \cap D),$$

where the three events inside the parentheses are mutually exclusive. Hence,

$$\begin{aligned} \pi(n) &= \text{prob}(R|A_0 = n) \\ &= \text{prob}\big((R \cap W)|A_0 = n\big) + \text{prob}\big((R \cap L)|A_0 = n\big) + \text{prob}\big((R \cap D)|A_0 = n\big) \end{aligned}$$

Note that

$$\text{prob}((X \cap Y)|Z) = \frac{\text{prob}(X \cap Y \cap Z)}{\text{prob}(Z)} = \frac{\text{prob}(X|(Y \cap Z))\text{prob}(Y \cap Z)}{\text{prob}(Z)} = \text{prob}(X|(Y \cap Z))\text{prob}(Y|Z)$$

Using this in the previous relation, we find

$$\begin{aligned} \pi(n) &= \text{prob}\big(R|(W \cap A_0 = n)\big)\text{prob}(W|A_0 = N) + \text{prob}\big(R|(L \cap A_0 = n)\big)\text{prob}(L|A_0 = N) + \\ &+ \text{prob}\big(R|(D \cap A_0 = n)\big)\text{prob}(D|A_0 = N) \\ &= p\,\text{prob}(R|A_1 = n+1) + q\,\text{prob}(R|A_1 = n-1) + r\,\text{prob}(R|A_1 = n) \\ &= p\,\text{prob}(R|A_0 = n+1) + q\,\text{prob}(R|A_0 = n-1) + r\,\text{prob}(R|A_0 = n). \end{aligned}$$

The last step follows from the fact that wins, losses, and ties are independent of both the current fortune and the time step we are at. Now we arrive at

$$\pi(n) = p\pi(n+1) + q\pi(n-1) + r\pi(n) \implies p\pi(n+1) - (p+q)\pi(n) + q\pi(n-1) = 0,$$

with $\pi(0) = 1$ and $\pi(N) = 0$. This is a second-order difference equation. Assuming $\pi(n) = \lambda^n$, we find

$$p\lambda^2 - (p+q)\lambda + q = 0 \implies \lambda = 1, \frac{q}{p}$$

i. If $p \neq q$, then

$$\pi(n) = C_1 + C_2 \left(\frac{q}{p}\right)^n$$

Using the boundary conditions $\pi(0) = 1$ and $\pi(N) = 0$, we get

$$C_2 = \frac{1}{1 - \left(\frac{q}{p}\right)^N}, \; C_1 = -\frac{\left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N} \implies \pi(n) = \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N}$$

Note that the probability of reaching fortune N starting with n rupees is therefore

$$\pi^c(n) = 1 - \pi(n) = \frac{1 - \left(\frac{q}{p}\right)^n}{1 - \left(\frac{q}{p}\right)^N}$$

a. $\underline{p > q}$: This is an example of super-fair betting. For example, consider $p = 0.51, q = 0.49, r = 0$, and $N = 100$. In this case, we find $\pi(10) \approx 0.664,\ \pi(50) \approx 0.119,\ \pi(75) \approx 0.032$. These are the probabilities of becoming broke. By subtracting these probabilities from 1, we get the probabilities of reaching the fortune $N = 100$.

If we let $N \to \infty$, then the gambler plays to become infinitely rich or broke. In this case, $\pi(n) \to \left(\frac{q}{p}\right)^n$ and $\pi^c(n) \to 1 - \left(\frac{q}{p}\right)^n$. With $p = 0.51, q = 0.49$, we find $\pi^c(10) \to 0.330,\ \pi^c(50) \to 0.865$ and $\pi^c(75) \to 0.95$. Thus, there is a positive probability that the gambler will never be ruined, but will instead become infinitely rich.

b. $\underline{p < q}$: This is an example of sub-fair betting. We rewrite $\pi(n)$ as

$$\pi(n) = \frac{\left(\frac{p}{q}\right)^{N-n} - 1}{\left(\frac{p}{q}\right)^{N} - 1}$$

For example, consider $q = 0.51, p = 0.49, r = 0$, and $N = 100$. In this case, $\pi(10) \approx 0.991,\ \pi(50) \approx 0.881$ and $\pi(75) \approx 0.644$. We also see that $\pi(n) \to 1$ as $N \to \infty$. Thus, if the gambler plays to become infinitely rich, he certainly will be broke in a sub-fair betting.

ii. If $p = q$ (i.e., fair betting), then $\lambda = 1, 1$ and hence $\pi(n) = C_1 + C_2 n$. Using the boundary conditions, we find $C_1 = 1$ and $C_2 = -\frac{1}{N}$ and hence

$$\pi(n) = 1 - \frac{n}{N}$$

If $N \to \infty$, then $\pi(n) \to 1$ and thus the gambler surely will be ruined if he plays to become infinitely rich.

Bold betting strategy: Instead of betting 1 rupee, the gambler may bet s rupees to become broke or arrive at N. Here, n and N must be divisible by s so that the amount of money reaches either 0 or N. For example, with $N = 200$ and $n = 100$, the value of s could be $1, 2, 5, 10, 20, 25, 50, 100$.

Changing the stakes means the unit of fortune has changed. Therefore, the gambler with 100 rupees determined to win 100 rupees planning with a bet of 10 rupees should have the same probability of ruin of a gambler with 10 rupees planning to win 10 with 1 rupee bet. Let $\Pi_s(n)$ be the probability of ruin starting with n rupees with a bet size of s rupees to get N rupees. Then

$$\begin{aligned}\Pi_s(n) &\equiv \text{probability of ruin with bet size 1 rupee to get } \frac{N}{s} \text{ rupees starting with } \frac{n}{s} \text{ rupees} \\ &= \frac{\left(\frac{q}{p}\right)^{n/s} - \left(\frac{q}{p}\right)^{N/s}}{1 - \left(\frac{q}{p}\right)^{N/s}} = \frac{\left(\frac{p}{q}\right)^{(N-n)/s} - 1}{\left(\frac{p}{q}\right)^{N/s} - 1}\end{aligned}$$

a. Consider the sub-fair betting with $q = 0.51, p = 0.49, r = 0$, and $N = 100$. With $s = 10$, we compute

$$\Pi_{10}(10) = \frac{\left(\frac{p}{q}\right)^9 - 1}{\left(\frac{p}{q}\right)^{10} - 1} \approx 0.917$$

Recall that we have previously calculated $\pi(10) \approx 0.991$. Thus, the probability of success with a 1 rupee bet is $\approx 1 - 0.991 \approx 0.009$. On the other hand, the probability of success with a 10 rupee bet is $\approx 1 - 0.917 \approx 0.083$. Thus, the probability of success with a 10 rupee bet is more than 9 times that of a 1 rupee bet. Similarly,

$$\Pi_{10}(50) = \frac{\left(\frac{p}{q}\right)^5 - 1}{\left(\frac{p}{q}\right)^{10} - 1} \approx 0.55,$$

and thus the probability of success with 10 rupees bet is ≈ 0.45. On the other hand, $\pi(50) \approx 0.881$ and thus the probability of success with a 1 rupee bet is ≈ 0.119. Again, we see that it is better to bet boldly in a sub-fair betting.

b. Now consider the super-fair betting with $p = 0.51, q = 0.49, r = 0$, and $N = 100$. We find $\Pi_{10}(10) \approx 0.881$ and $\pi(10) \approx 0.664$. Now, the probability of success with a 1 rupee bet is ≈ 0.336. On the other hand, the probability of success with a 10 rupee bet is ≈ 0.119. Thus, it is better to play cautiously when betting is super-fair.