Stochastic models: Birth and death process II

Pure death process: Now, we consider the pure death process in a population. The death rate in each state is only dependent on the current state. Let $X(t)$ be the number of individuals alive up to and including time t, given that the initial size is n_0. We are interested in

$$p_n(t) = \text{prob}[X(t) = n \big| X(0) = n_0], \qquad n = 0, 1, 2, \cdots, n_0.$$

The following are the assumptions underlying the pure death process:

i. Given $X(t) = n$, the probability of a new death occurring in $(t, t+h]$ is $\mu_n\, h + o(h)$, where μ_n is a function of n only.

ii. The probability that more than one death will occur in the time interval $(t, t+h]$ is $o(h)$

iii. Once $X(t) = 0$, the process stays at 0 permanently.

Hence, the probability of no death in $(t, t+h]$ is $1 - \mu_n h - o(h)$.
First, consider the case of $1 \leq n \leq n_0 - 1$:

$$\begin{aligned} p_n(t+h) &= p_n(t) \times \text{prob}\{\text{no death in } (t, t+h]\} + p_{n+1}(t) \times \text{prob}\{\text{one death in } (t, t+h]\} + \\ &+ \sum_{i \geq 2}^{n_0 - n} p_{n+i}(t) \times \text{prob}\{\ i \text{ death in } (t, t+h]\} \\ &= p_n(t)[1 - \mu_n h - o(h)] + p_{n+1}(t)[\mu_{n+1} h + o(h)] + o(h) \end{aligned}$$

Note that the sum in the last term contributes $o(h)$ only when $n \leq n_0 - 2$. Rearranging and dividing both sides by h, we get

$$\frac{p_n(t+h) - p_n(t)}{h} = -\mu_n p_n(t) + \mu_{n+1} p_{n+1}(t) + \frac{o(h)}{h}$$

As $h \to 0$, we find

$$\frac{dp_n(t)}{dt} = -\mu_n p_n(t) + \mu_{n+1} p_{n+1}(t)$$

For $n = 0$, we have

$$\begin{aligned} p_0(t+h) &= p_0(t) \times \text{prob}\{\text{no death in } (t, t+h]\} + p_1(t) \times \text{prob}\{\text{one death in } (t, t+h]\} + \\ &+ \sum_{i \geq 2}^{n_0} p_i(t) \times \text{prob}\{i \text{ death in } (t, t+h]\} \\ &= p_0(t) + p_1(t)[\mu_1 h + o(h)] + o(h) \end{aligned}$$

Note that if the number of individuals alive up to and including time t is 0, the probability of no death in $(t, t+h]$ is 1. Rearranging and dividing both sides by h, we get

$$\frac{dp_0(t)}{dt} = \mu_1 p_1(t).$$

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For $n = n_0$, we have

$$\begin{aligned} p_{n_0}(t+h) &= p_{n_0}(t) \times \text{prob}\{\text{no death in } (t, t+h]\} \\ &= p_{n_0}(t)[1 - \mu_{n_0} h - o(h)] \end{aligned}$$

Rearranging and dividing both sides by h, we get

$$\frac{dp_{n_0}(t)}{dt} = -\mu_{n_0} p_{n_0}(t).$$

Hence, the governing equations become

$$\begin{aligned} \frac{dp_{n_0}(t)}{dt} &= -\mu_{n_0} p_{n_0}(t), \\ \frac{dp_n(t)}{dt} &= -\mu_n p_n(t) + \mu_{n+1} p_{n+1}(t), \qquad 1 \le n \le n_0 - 1, \\ \frac{dp_0(t)}{dt} &= \mu_1 p_1(t). \end{aligned}$$

The initial conditions are $p_{n_0}(0) = 1$ and $p_n(0) = 0$ for $n = 0, 1, 2, \cdots, n_0 - 1$.

Linear death process: In this case, we have $\mu_n = \mu\, n$. Now, solving the first equation, we get

$$p_{n_0}(t) = e^{-\mu\, n_0\, t} = \left(e^{-\mu t}\right)^{n_0}$$

For $n = n_0 - 1$, we have

$$\frac{p_{n_0-1}(t)}{dt} + (n_0 - 1)\mu p_{n_0-1}(t) = n_0\, \mu\, p_{n_0}(t)$$

Solving this, we find

$$p_{n_0-1}(t) = n_0 \left(e^{-\mu t}\right)^{n_0-1} (1 - e^{-\mu t}) = \binom{n_0}{n_0 - 1} \left(e^{-\mu t}\right)^{n_0-1} (1 - e^{-\mu t})^{n_0-(n_0-1)}$$

It can be proved by induction that

$$p_n(t) = \binom{n_0}{n} \left(e^{-\mu t}\right)^n (1 - e^{-\mu t})^{n_0-n}, \qquad 1 \le n \le n_0 - 1.$$

Hence,

$$p_1(t) = \binom{n_0}{1} e^{-\mu t} (1 - e^{-\mu t})^{n_0-1} = n_0 e^{-\mu t} (1 - e^{-\mu t})^{n_0-1}$$

Now

$$\frac{dp_0(t)}{dt} = \mu p_1(t) = \mu n_0 e^{-\mu t} (1 - e^{-\mu t})^{n_0-1}$$

Integrating from 0 to t, we find

$$p_0(t) = (1 - e^{-\mu t})^{n_0}$$

From the solution, it is clear that we can write

$$p_n(t) = \binom{n_0}{n} \left(e^{-\mu t}\right)^n 1 - e^{-\mu t})^{n_0-n}, \qquad n = 0, 1, 2, \cdots, n_0.$$

Hence, $X(t)$, which denotes the number of individuals alive up to and including time t given the initial size n_0, follows Binomial(n_0, p), where $p = e^{-\mu t}$.

Birth and death process: When a birth occurs, the process goes from state n to $n+1$. Similarly, when death occurs, the process goes from state n to state $n-1$. It is assumed that the birth and death events are independent. The birth-and-death process is characterised by the birth rate λ_n and death rate μ_n, which vary according to the state n of the process. Let $X(t)$ be the number of individuals up to and including time t, given that there are initially n_0 individuals. We are interested in

$$p_n(t) = \text{prob}[X(t) = n \big| X(0) = n_0], \qquad n = 0, 1, 2, \cdots, n_0, n_0 + 1, \cdots$$

The following are the assumptions underlying the birth-and-death process:

i. The probability of $X(t+h) = n+1$ given that $X(t) = n$ is $\lambda_n h + o(h)$

ii. The probability of $X(t+h) = n-1$ given that $X(t) = n$ is $\mu_n h + o(h)$

iii. The probability that more than one birth or more than one death will occur in the time interval $(t, t+h]$ is $o(h)$

Hence, the probability of $X(t+h) = n$ given that $X(t) = n$ is $1 - \lambda_n h - \mu_n h - o(h)$.

For $n \geq 1$, we find

$$\begin{aligned} p_n(t+h) &= p_n(t) \times \text{prob}\{X(t+h) = X(t)\} + p_{n-1}(t) \times \text{prob}\{X(t+h) = X(t)+1\} + \\ &+ p_{n+1}(t) \times \text{prob}\{X(t+h) = X(t)-1\} + \sum_{i\geq 2} p_{n-i}(t) \times \text{prob}\{X(t+h) = X(t)+i\} + \\ &+ \sum_{i\geq 2} p_{n+i}(t) \times \text{prob}\{X(t+h) = X(t)-i\} \\ &= p_n(t)[1 - \lambda_n h - \mu_n h - o(h)] + p_{n-1}(t)[\lambda_{n-1} h + o(h)] + p_{n+1}(t)[\mu_{n+1} h + o(h)] + o(h) \end{aligned}$$

Rearranging and dividing both sides by h, we get

$$\frac{p_n(t+h) - p_n(t)}{h} = -\lambda_n p_n(t) - \mu_n p_n(t) + \lambda_{n-1} p_{n-1}(t) + \mu_{n+1} p_{n+1}(t) + \frac{o(h)}{h}$$

Taking $h \to 0$, we find

$$\frac{dp_n(t)}{dt} = -(\lambda_n + \mu_n) p_n(t) + \lambda_{n-1} p_{n-1}(t) + \mu_{n+1} p_{n+1}(t), \qquad n = 1, 2, 3, \cdots$$

For $n = 0$, we find

$$\begin{aligned} p_0(t+h) &= p_0(t) \times \text{prob}\{X(t+h) = X(t)\} + p_1(t) \times \text{prob}\{X(t+h) = X(t)-1\} + \\ &+ \sum_{i\geq 2} p_{n+i}(t) \times \text{prob}\{X(t+h) = X(t)-i\} \\ &= p_0(t)[1 - \lambda_0 h - o(h)] + p_1(t)\mu_1 h + o(h) \end{aligned}$$

To understand the first expression on the right-hand side, note that though the population size is 0 up to and including time t, there is a possibility of birth in the interval $(t, t+h]$. For

example, think about plants growing from germinating seeds. There may not be plants in $(0, t]$, but a new plant may come up in $(t, t + h]$. Rearranging and dividing both sides by h, we get

$$\frac{p_0(t+h) - p_0(t)}{h} = -\lambda_0 p_0(t) + \mu_1 p_1(t) + \frac{o(h)}{h}$$

Taking $h \to 0$, we find

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) + \mu_1 p_1(t)$$

These two equations

$$\begin{aligned} \frac{dp_0(t)}{dt} &= -\lambda_0 p_0(t) + \mu_1 p_1(t) \\ \frac{dp_n(t)}{dt} &= -(\lambda_n + \mu_n) p_n(t) + \lambda_{n-1} p_{n-1}(t) + \mu_{n+1} p_{n+1}(t), \qquad n = 1, 2, 3, \cdots \end{aligned}$$

are referred to as the basic difference-differential equations for the birth-and-death process. The initial conditions are $p_{n_0}(0) = 1$ and $p_n(0) = 0$ for $n \neq n_0$. To "close" (or truncate/bound) a stochastic birth-and-death process, we set boundary conditions or limit the state space so that the infinite or open-ended system becomes manageable.

Since explicit solutions for $p_n(t)$ are not readily available, the steady-state solutions are important. The steady state implies that the solutions $p_n(t)$ are independent of time after a long time. Hence, $p_n(t)$ are constants and $p_n'(t) = 0$. Consequently, under steady state assumption, we have $P_n = \lim_{t\to\infty} p_n(t)$, and

$$\begin{aligned} -\lambda_0 P_0 + \mu_1 P_1 &= 0, \qquad n = 0, \\ -(\lambda_n + \mu_n) P_n + \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} &= 0, \qquad n = 1, 2, 3, \cdots \end{aligned}$$

Solving the equation for $n = 0$, we find

$$P_1 = \frac{\lambda_0}{\mu_1} P_0$$

For $n = 1$, we get

$$\begin{aligned} (\lambda_1 + \mu_1) P_1 &= \lambda_0 P_0 + \mu_2 P_2 = \mu_1 P_1 + \mu_2 P_2 \qquad \text{[using the relation for } n = 0 \\ \Longrightarrow \qquad P_2 &= \frac{\lambda_1}{\mu_2} P_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0 \end{aligned}$$

Proceeding in this manner, we find

$$P_n = \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_1 \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1} P_0 = P_0 \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}, \qquad n \geq 1.$$

To find P_0, we use the definition of probability to find

$$\sum_{n=0}^{\infty} P_n = 1 \implies P_0 \left(1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \right) = 1 \implies P_0 = \left(1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \right)^{-1}.$$

For a stationary or steady-state distribution to exist, this infinite sum must converge.