Lake pollution model I

Let $V(t)$ be the volume of a lake and $c(t)$ (mass/vol) be the concentration of a pollutant at time t. Let $Q_{in}(t)$ (vol/time) and $Q_{out}(t)$ (vol/time) be the volumetric flow into and out of the lake, respectively. Further, suppose that the pollutant may decay in the lake due to natural processes, and this decay is proportional to the amount of the pollutant in the lake. Let $M(t)$ be the mass of the pollutant in the lake at time t. We assume that the pollutant enters the lake only through the inflow. The governing equation for the pollutant concentration follows from mass conservation:

$$\frac{dM(t)}{dt} = Q_{in}(t)c_{in}(t) - Q_{out}(t)c_{out}(t) - kM(t),$$

where $c_{in}(t)$ and $c_{out}(t)$ are the pollutant concentration in the inflow and outflow, and $k > 0$ is the proportionality constant for the pollutant decay in the lake. Since $M(t) = V(t)c(t)$, the above equation can be written as

$$\frac{d}{dt}\big(c(t)V(t)\big) = Q_{in}(t)c_{in}(t) - Q_{out}(t)c_{out}(t) - kV(t)c(t).$$

Note that if $Q_{in}(t)$ and $Q_{out}(t)$ are equal, then $V(t) = V$ is a constant. We made further assumptions to simplify the model:

i. The inflow and outflow rates are constant, i.e., $Q_{in}(t) = Q_{out}(t) = Q$.

ii. We assume that the lake is well-mixed, and thus the outflow has the same concentration as the lake, i.e., $c_{out}(t) = c(t)$.

Now the model becomes

$$\frac{dc}{dt} = \frac{Q}{V}c_{int}(t) - \frac{Q}{V}c - kc,$$

where c has to be understood as a function of t. Let $T = \frac{V}{Q}$ be the drainage time or residence time of a pollutant. It is the mean time a particular pollutant spends in the lake. Then we get

$$\frac{dc}{dt} + \frac{c}{T} + kc = \frac{c_{int}(t)}{T}.$$

This is to be solved subject to the initial condition $c(t = 0) = c_0$.
Example: Choose $k = 0$ and a *short-duration* input:

$$c_{in}(t) = \begin{cases} c_I, & 0 \le t \le t_I, \\ 0, & t > t_I. \end{cases}$$

The governing equation is

$$\frac{dc}{dt} + \frac{c}{T} = \frac{c_{int}(t)}{T}, \qquad c(t = 0) = c_0.$$

For $0 \le t \le t_I$, we find

$$ce^{t/T} - c_0 = \frac{c_I}{T}\int_0^t e^{s/T} ds \implies c = (c_0 - c_I)e^{-t/T} + c_I.$$

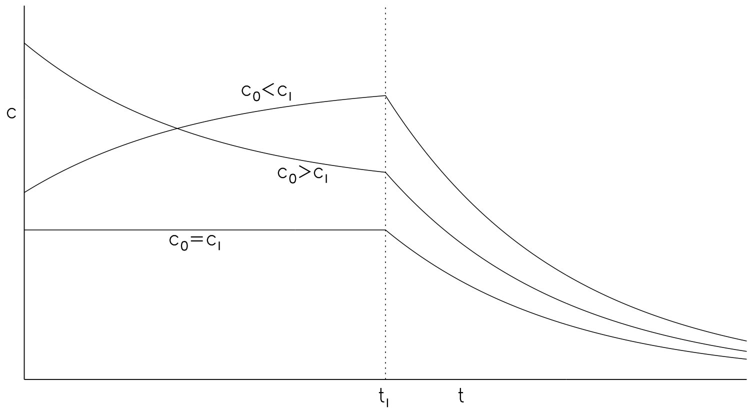
For $t \geq t_I$, we find

$$ce^{t/T} - c_0 = \frac{c_I}{T}\int_0^{t_I} e^{s/T} ds + \frac{1}{T}\int_{t_I}^{t} 0\, ds \implies c = (c_0 - c_I)e^{-t/T} + c_I e^{-(t-t_I)/T}.$$

From the solution, we observe that

$$\frac{dc}{dt} = \begin{cases} \frac{c_I - c_0}{T} e^{-t/T}, & 0 < t < t_I, \\ -\frac{c}{T}, & t > t_I. \end{cases}$$

Clearly, $c(t)$ decreases for $t > t_I$ for any value of c_0 and c_I. In the interval $(0, t_I)$, it decreases if $c_0 > C_I$, increases if $c_0 < c_I$ and remains constant c_0 if $c_0 = c_I$. Also, note that if t_I is large, then $c(t) \to c_I$ as $t \to t_I$. Figure shows that characteristics of the solutions just discussed.



<u>Example</u>: Choose $k = 0$ and a *step-time* input:

$$c_{in}(t) = \begin{cases} 0, & 0 \leq t \leq t_S, \\ c_S, & t > t_S. \end{cases}$$

The governing equation is

$$\frac{dc}{dt} + \frac{c}{T} = \frac{c_{int}(t)}{T}, \qquad c(t=0) = c_0.$$

For $0 \leq t \leq t_S$, we find

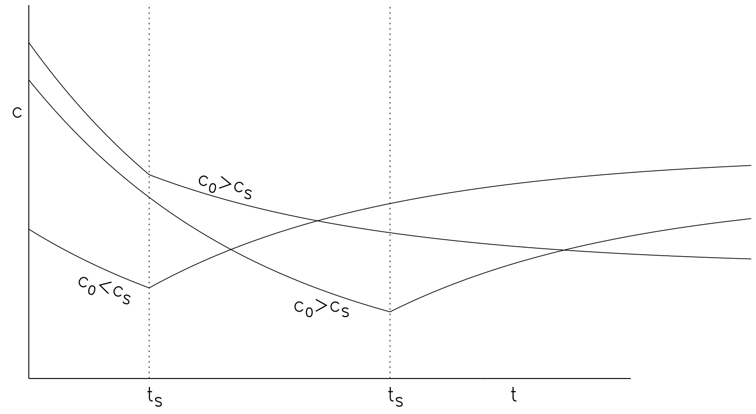
$$ce^{t/T} - c_0 = \frac{1}{T}\int_0^t 0 \cdot e^{s/T} ds \implies c = c_0 e^{-t/T}$$

For $t \geq t_S$, we find

$$ce^{t/T} - c_0 = \frac{1}{T}\int_0^{t_I} 0 \cdot e^{s/T}\, ds + \frac{c_S}{T}\int_{t_S}^{t} e^{s/T} ds \implies c = c_0 e^{-t/T} + c_S\left(1 - e^{-(t-t_S)/T}\right).$$

From the solution, we observe that $c(t) \to c_S$ as $t \to \infty$. Differentiating the solution, we find

$$\frac{dc}{dt} = \begin{cases} -\frac{c_0}{T} e^{-t/T}, & 0 < t < t_S, \\ \frac{1}{T}\left(-c_0 e^{-t_S/T} + c_S\right) e^{-(t-t_S)/T}, & t > t_S. \end{cases}$$



Clearly, the concentration always decreases in $(0, t_S)$. If $c_S > c_0$, then $c_S > c_0 \mathrm{e}^{-t_S/T}$ and hence the concentration increases for $t > t_s$ as shown in the figure. If $c_0 > c_S$, then two cases arise:

i. If $c(t_S-) > c_S$, then the concentration also decreases for $t > t_S$ (see top curve in the figure).

i. If $c(t_S-) < c_S$, then the concentration increases for $t > t_S$ (see the curve with the largest t_S in the figure).