Lake pollution model II

Example: We may take a *sinusoidal input* for the pollutant concentration:

$$c_{in}(t) = c_M \left(1 + \epsilon \sin\left(\frac{2\pi t}{T_p}\right)\right), \qquad 0 \le \epsilon \le 1,$$

where T_p is the time period of the fluctuation. Here, c_M is the mean pollutant concentration. Note that $\epsilon > 1$ is not possible since concentration cannot be negative. For example, we may take $T_p = 24$ and measure time from 6 AM. Then the concentration is above the mean concentration from 6 AM to 6 PM and below the mean concentration from 6 PM to 6 AM. Such an input may model the situation in which a plant releases more pollutants into the inflow during the daytime than at night. This can be solved analytically, too.

Modification of the model: The original model using only the well-mixed assumption is

$$\frac{d}{dt}\big(c(t)V(t)\big) = Q_{in}(t)c_{in}(t) - Q_{out}(t)c(t) - kV(t)c(t).$$

Case I: If $Q_{in}(t) = Q_{out}(t) = Q(t)$, then the inflow and outflow rates are equal, but they are functions of time. Then the model equation becomes

$$\frac{dc}{dt} + \left(\frac{Q(t)}{V} + k\right)c = \frac{Q(t)}{V}c_{in}(t)$$

For example, we may take

$$c_{in}(t) = c_M \left(1 + \epsilon_1 \cos\left(\frac{2\pi t}{T_p}\right)\right), \quad Q(t) = Q_M \left(1 - \epsilon_2 \cos\left(\frac{2\pi t}{T_p}\right)\right), \qquad 0 \le \epsilon_1, \epsilon_2 \le 1,$$

where c_M and Q_M are the mean pollutant concentration and mean flow rate. Usually, a low flow rate is associated with high pollutant concentrations, and vice versa; the above describes this situation.

Case II: If the flow rates are time-dependent and different, then the volume is a function of time. In this case, we need to solve

$$\frac{d}{dt}\big(c(t)V(t)\big) = Q_{in}(t)c_{in}(t) - Q_{out}(t)c(t) - kV(t)c(t), \quad \frac{dV}{dt} = Q_{in}(t) - Q_{out}(t),$$

subject to $c(0) = c_0$ and $V(0) = V_0$.

If the flow rates are constant but different, then the above equations simplify to

$$\frac{d}{dt}\big(c(t)V(t)\big) = Q_{in}c_{in}(t) - Q_{out}c(t) - kV(t)c(t), \quad V(t) = V_0 + (Q_{in} - Q_{out})t,$$

subject to $c(0) = c_0$. Note that Q_{in} and Q_{out} are constant.

Time for lake purification: Here, we consider the simplified model in which the lake is well-mixed and inflow-outflow rates are equal and constant. Assuming $k = 0$, the equation is

$$\frac{dc}{dt} + \frac{c}{T} = \frac{c_{int}(t)}{T}, \qquad c(0) = c_0.$$

Suppose the pollutant in the inflow is stopped at $t = 0$. Then the above equation reduces to

$$\frac{dc}{dt} + \frac{c}{T} = 0, \qquad c(0) = c_0,$$

whose solution is $c = c_0 \mathrm{e}^{-t/T}$. Clearly, $c(t) \to 0$ as $t \to \infty$. This implies that the lake can be purified in a very long time. Instead, let us find the time $t_{0.05}$ by which that pollutant concentration becomes 5% of the initial concentration. This implies

$$\frac{c_0}{20} = c_0 \mathrm{e}^{-t_{0.05}/T} \implies t_{0.05} = T \ln 20 \approx 3T.$$

Thus, the pollutant reduces to 5% of the initial value in approximately three times the drainage time. Note that we have assumed a well-mixed assumption. If the pollutant is poorly mixed, then the purification process will be prolonged. Hence, the above time can be considered a lower bound.

A case study: Consider a lake A with constant volume $V_A = V$ and a lake B with constant volume $V_B = 3V$. Both lakes have the same constant inflow and outflow rates Q. Lake B receives $\frac{3}{4}$-th inflow from lake A and $\frac{1}{4}$-th inflow from other sources like rain, other streams, etc. Suppose that lake A is more polluted than lake B, and the inflow from other sources to lake B is pollutant-free. Further, the inflow to lake A becomes pollutant-free at $t = 0$. We need to find the time $t_{0.05}$ for lake B.

The drainage times are respectively $T_A = \frac{V}{Q} = T$ and $T_B = \frac{3V}{Q} = 3T$. Since the inflow to lake A is pollutant-free, the governing equation for lake A is

$$V_A \frac{dc_A}{dt} + Qc_A = 0 \implies \frac{dc_A}{dt} + \frac{c_A}{T} = 0, \qquad c_A(0) = c_{0A}.$$

Its solution is

$$c_A(t) = c_{0A} \mathrm{e}^{-t/T}$$

For lake B, we know that the incoming flow from lake A has pollutant concentration $c_A(t)$ and the other inflow has zero pollutant. Hence,

$$V_B \frac{dc_B}{dt} = \frac{3}{4} Q c_A(t) + \frac{1}{4} Q \cdot 0 - Q c_B(t), \qquad c_B(0) = c_{0B}.$$

Simplifying, we get

$$\frac{dc_B}{dt} + \frac{c_B}{T_B} = \frac{3c_{0A}}{4T_B} \mathrm{e}^{-t/T}, \qquad c_B(0) = c_{0B}.$$

Integrating we find

$$c_B e^{t/T_B} - c_{0B} = \frac{3c_{0A}}{4T_B} \int_0^t \mathrm{e}^{(1/T_B - 1/T)s} ds \implies c_B e^{t/T_B} = c_{0B} + \frac{3c_{0A}}{4T_B} \frac{1}{\frac{1}{T_B} - \frac{1}{T}} \left(\mathrm{e}^{(1/T_B - 1/T)t} - 1 \right),$$

which on simplification becomes

$$c_B e^{t/T_B} = c_{0B} + \frac{3c_{0A} T}{4(T - T_B)} \left(\mathrm{e}^{(1/T_B - 1/T)t} - 1 \right).$$

Using $T = \frac{T_B}{3}$, we find

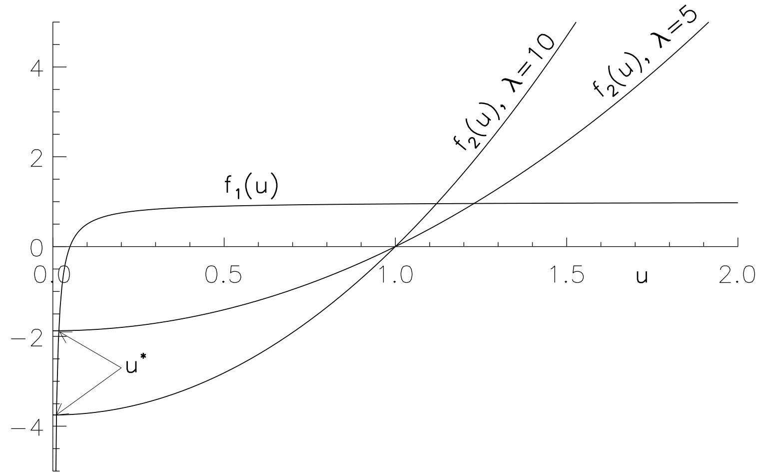
$$c_B e^{t/T_B} = c_{0B} - \frac{3c_{0A}}{8} \left(\mathrm{e}^{-2t/T_B} - 1 \right).$$

We find $t_{0.05}$ so that $c_B(t_{0.05}) = \frac{c_{0B}}{20}$. Putting this in the above equation and simplifying, we get

$$\frac{1}{20}e^{t_{0.05}/T_B} = 1 - \frac{3\lambda}{8}\left(\mathrm{e}^{-2t_{0.05}/T_B} - 1\right),$$

where $\lambda = \frac{c_{0A}}{c_{0B}}$. Since the lake A is more polluted than lake B, this implies $\lambda > 1$. Let $u = e^{-t_{0.05}/T_B}$, then $0 < u < 1$ and the above becomes

$$\frac{1}{20u} = 1 - \frac{3\lambda}{8}(u^2 - 1) \implies 1 - \frac{1}{20u} = \frac{3\lambda}{8}(u^2 - 1)$$



To find the root which must be positive and less than unity, we plot $f_1(u) = 1 - \frac{1}{20u}$ and $f_2(u) = \frac{3\lambda}{8}(u^2 - 1)$ in the right half. Note that $f_1(u)$ intersects the u-axis at $u = \frac{1}{20}$, $f_1(u) \to -\infty$ as $u \to 0+$, and $f_1(u) \to 1$ as $u \to \infty$. Further, $f_1(u)$ is concave down and strictly increasing (like the left branch of the parabola $y = -x^2$). On the other hand, $f_2(u)$ intersects the u-axis at $u = 1$ and $f_2(0) = -\frac{3\lambda}{2} < 0$, and $f_2(u) \to \infty$ as $u \to \infty$. Further, $f_2(u)$ is strictly increasing for $u > 0$, and it is concave up (like the right branch of the parabola $y = x^2$). These two graphs show that there exists a unique root u^* that satisfies $0 < u^* < 1$. When λ increases, then u^* decreases and hence $(u^*)^2$ is negligible for large λ. Supposing that λ is large, we find

$$1 - \frac{1}{20u^*} \approx -\frac{3\lambda}{8} \implies \frac{1}{u^*} \approx 20 + \frac{15\lambda}{2} \implies e^{t_{0.05}/T_B} \approx 20 + \frac{15\lambda}{2} \implies t_{0.05} \approx \ln\left(\frac{5}{2}(8 + 3\lambda)\right) T_B$$

This formula even yields acceptable values for small λ, as the following table makes clear. In the second column, $1/u^*$ is calculated using MATLAB from the cubic polynomial.

λ	$1/u^*$ (using MATLAB)	$1/u^* \approx 20 + \frac{15\lambda}{2}$
10	0.01053	0.01053
5	0.01739	0.01739
2	0.02858	0.02857
1.6	0.03126	0.03125