ESO 209: PROBABILITY & STATISTICS
Session: 2009-10 Semester: II
(Instructor: S. Mitra)

Problem Set 1

[1] Let $\Omega = \{1,2,3,4\}$. Check whether the following classes of sets are σ-fields or not:

$\mathcal{F}_1 = \{\phi, \{1,2\}, \{3,4\}\}$

$\mathcal{F}_2 = \{\phi, \Omega, \{1\}, \{2,3,4\}, \{1,2\}, \{3,4\}\}$

$\mathcal{F}_3 = \{\phi, \Omega, \{1\}, \{2\}, \{1,2\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}\}$

[2] Let $\mathcal{F}$ be a σ-field of subsets of the sample space Ω and let $A \in \mathcal{F}$ be fixed. Verify whether $\mathcal{F}_A = \{C : C = A \cap B, B \in \mathcal{F}\}$ is a σ-field of subsets of A or not.

[3] Let $\Omega = \{0, 1, 2, \ldots\}$. In each of the following cases check if P is a probability measure, where, for an event A,

(a) $P(A) = \sum_{x \in A} \frac{e^{-\lambda}\lambda^x}{x!}, \lambda > 0.$

(b) $P(A) = \sum_{x \in A} p\,(1-p)^x, 0 < p < 1.$

(c) $P(A) = \begin{cases} 1 & \text{if the number of elements in } A \text{ is finite} \\ 0 & \text{otherwise} \end{cases}$

In cases where your answer is in the affirmative, find $P(E), P(F), P(G)$, $P(E \cap F)$, $P(E \cup F)$, $P(F \cup G), P(E \cap G)$ and $P(F \cap G)$, where $E = \{x \in \Omega : x > 2\}$, $F = \{x \in \Omega : 0 < x < 3\}$ and $G = \{x \in \Omega : 3 < x < 6\}$.

[4] Let $\Omega = \Re$. In each of the following cases check if P is a probability measure, where, for an interval I,

(a) $P(I) = \int_I \frac{1}{2} e^{-|x|} dx$

(b) $P(I) = \int_I \frac{1}{\pi} \frac{1}{1+x^2} dx$

(c) $P(I) = \begin{cases} 0 & \text{if } I \subset (-\infty, 1) \\ \int_I \frac{dx}{2} & \text{if } I \subset [1, \infty) \end{cases}$

(d) $P(I) = \begin{cases} 1 & \text{if length of } I \text{ is finite} \\ 0 & \text{otherwise} \end{cases}$

In case where your answer is in the affirmative, find $P(x : x \geq 0)$.

[5] Describe the sample space when a coin is tossed (a) once, (b) three times, (c) n times and (d) an infinite number of times.

[6] A coin is tossed until for the first time the same result appear twice in succession. To an outcome requiring n tosses assign a probability 2^{-n}. Describe the sample space. Evaluate the probability of the following events:
(a) A= the experiment ends before the 6th toss;
(b) B= an even number of tosses are required;
(c) $A \cup B,\ A \cap B, A \cap B^C,\ A^C \cap B^C, A^C \cap B$

[7] Show that if a set function assumes value 1 at the empty set then it cannot be additive.

[8] Let the sample space Ω of a random experiment be an infinite set and suppose that A is an event if either A or A^C is a finite subset of Ω. Define $P(A) = 0\ or\ 1$, according as A is finite or not. Show that P is finitely additive but not countably additive.

[9] Show that the probability of exactly one of the events A or B occurring is $P(A) + P(A) - 2P(A \cap B)$.

[10] What is the probability that among k random digits
(a) 0 does not appear, (b) 1 does not appear, (c) neither 0 nor 1 does not appear and (d) at least one of the digits 0 or 1 does not appear.
Let A and B represent the events in parts (a) and (b), respectively. Express the other events in terms of A and B.

[11] Consider the sample space $\Omega = \{0,1,2,...\}$ and $\mathcal{F}$ the $\sigma-$field of subsets of Ω. To the elementary event $\{j\}$ assign the probability

$$P(\{j\}) = c\frac{2^j}{j!}, \quad j = 0,1,2,....$$

(a) Determine the constant c.
(b) Define the events A, B and C by
$A = \{j: 2 \le j \le 4\},\ B = \{j: j \ge 3\}, C = \{j: j \text{ is an odd integer}\}$.
Evaluate $P(A), P(B), P(C), P(A \cap B), P(A \cap C), P(B \cap C), P(A \cap B \cap C)$ and verify the formula for $P(A \cup B \cup C)$.

[12] Three tickets are drawn randomly without replacement from a set of tickets numbered 1 to 100. Find the probability that the number of selected tickets are in (a) arithmetic progression and (b) geometric progression.

[13] Three players A, B and C play a series of games, none of which can be drawn and their probability of winning any game are equal. The winner of each game scores 1 point and the series is won by the player who first scores 4 points. Out of the first three games A won 2 games and B won 1 game. Find the probability that C will win the series.

[14] A point P is randomly placed in a square with side of 1 cm. Find the probability that the distance from P to the nearest side does not exceed x cm.

[15] Let there be n people in a room and p denote the probability that there are no common birthdays. Find an approximate value of p for $n = 10$.

[16] Which is more probable to appear – (a) at least 1 ace in 4 throws of a die, or, (b) at least 1 double ace in 24 throws of a pair of dice?

[17] If n men, among whom are A and B, stand in a row, what is the probability that there will be exactly r men between A and B? What is this probability if they stand in a circle?

[18] Each packet of a certain cereal contains a small plastic model of one of the five different dinosaurs; a given packet is equally likely to contain any one of the five dinosaurs. Find the probability that someone buying six packets of the cereal will acquire models of three favorite dinosaurs.

[19] Suppose n cards numbered $1, 2, \ldots, n$ are laid out at random in a row. Let A_i denote the event that 'card i appears in the i^{th} position of the row', which is termed as a match. What is the probability of obtaining at least one match?

[20] In a town of $n+1$ inhabitants, a person tells a rumour to a second person, who in turn tells it to a third person, and so on. At each step the recipient of the rumour is chosen at random from the n people available. Find the probability that the rumour will be told r times without

(a) returning to the originator,

(b) being repeated to any person.

Do the same problem when at each step the rumour is told to a gathering of N randomly chosen people.