END SEMESTER EXAMINATION
PROBABILITY THEORY MTH309A
VENUE: L1 OROS
DATE AND TIME: APRIL 26, 2019. 16:00 TO 19:00 HRS

NAME:
ROLL NO:

General instructions:

- **ATTACH this question paper to your answer-script at the time of submission.**
- Each section below has specific instructions. Read them carefully.
- You can use your class notes. Searching in books/internet is not allowed.
- Maximum you can score: 40 (Answer as much as you can)
- Notations:
  - $\mathbb{B}_{\mathbb{R}^n}$: the Borel σ-field on $\mathbb{R}^n$, generated by the open sets.
  - $(\Omega, \mathcal{F}, \mathbb{P})$: a probability space.
  - $\mathbb{Q}$: the set of rational numbers
  - $\mathbb{Z}$: the set of integers

1. Section A

Question 1. **Tick(✓) the correct option on this question paper** (do not write on the answer-script). Each question has only one correct answer. You do not get any credit for the rough work. You do not get any credit for illegible answers.

(i) Identify which one of the following classes of sets does NOT generate $\mathbb{B}_{\mathbb{R}^2}$. [3]
- $\{(-\infty, x] \times [-\infty, y] : x, y \in \mathbb{R}\}$.
- $\{B(\binom{x}{y}, r) : \binom{x}{y} \in \mathbb{Z}^2, r \in \mathbb{Q}, r > 0\}$, where $B(\binom{x}{y}, r)$ denotes the open ball centred at $\binom{x}{y}$ with radius r.
- $\{[x, x+1] \times [y, y+1] : x, y \in \mathbb{R}\}$.
- $\{[x, \infty) \times (y, \infty) : x, y \in \mathbb{R} \setminus \mathbb{Q}\}$.

(ii) Let $\{A_n\}_{n=1}^\infty$ be a sequence of sets in $\mathcal{F}$ with the property that $\mathbb{P}(A_n) = 1, \forall n$. Then the value of $\mathbb{P}(\bigcap_{n=1}^\infty A_n)$ is [2]
- is 0.
- is 1.
- is $\frac{1}{2}$.
- can be any value in $(0, 1)$.

(iii) Let $\{X_n\}_{n=1}^\infty$ be a sequence of independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with $X_n \sim N(1, \frac{1}{n^2}), \forall n$. Then $\sum_{n=1}^\infty (1 - X_n)$ [3]
- converges a.s..
- diverges a.s..
- converges with probability $\frac{1}{2}$.
- can not be determined from the given information.

(iv) Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of probability measures (on $\mathbb{R}$) converging weakly to $\frac{1}{3}\delta_1 + \frac{2}{3}\delta_{-1}$. Then the limit of characteristic functions $\lim_{n\to\infty} \Phi_{\mu_n}(t)$ [2]
- does not exist
- $\frac{1}{3}\exp(it) + \frac{2}{3}\exp(-it), \forall t \in \mathbb{R}$.
- $\exp(-\frac{1}{3}it), \forall t \in \mathbb{R}$.
- $\exp(\frac{1}{3}it - \frac{1}{3}t^2), \forall t \in \mathbb{R}$.

(v) Let F and G be the distribution functions of real valued random variables X and Y respectively. Identify which of the following is NOT a distribution function of some random variable. [2]
- F^2.
- FG.
- $\frac{1}{2}(F + G)$.
- $\min\{F - G, G - F\}$.

(vi) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued, measurable functions defined on a measure space $(\Omega, \mathcal{F}, \mu)$. Suppose f_n's converge pointwise to a function f. Which one of the following conditions does <u>NOT</u> imply the equality $\lim_{n\to\infty} \int_\Omega f_n(\omega)\, d\mu(\omega) = \int_\Omega f(\omega)\, d\mu(\omega)$? [3]

- The functions f_n are bounded independent of n and the measure μ is finite.
- The measure μ is a probability measure and there exists a non-negative measurable function g on Ω such that $|f_n(\omega)| \leq g(\omega), \forall n, \omega$ and $\int_\Omega g(\omega)^2\, d\mu(\omega) < \infty$.
- The measure μ is infinite and f_n's are non-negative and increasing in n.
- The measure μ is finite and the functions f_n are integrable.

(vii) Consider $\mathcal{C} := \{(a, b) : -\infty \leq a < b \leq \infty\}$. Then $\mathcal{C}$ is closed under [3]

- complementation.
- countable unions.
- countable increasing unions.
- countable intersections.

(viii) Let $A \in \mathcal{F}$ be such that it is independent of itself. Then $\mathbb{P}(A)(1 - \mathbb{P}(A))$ [2]

- is $\frac{1}{4}$.
- is 0.
- is $\frac{1}{3}$.
- can be any value in $\left(0, \frac{1}{4}\right)$.

2. Section B

Instruction: You may use any result proved in class.

<u>*Question*</u> 2. Let μ and ν be two measures defined on the measurable space $(\Omega, \mathcal{F})$ such that $\mu(\Omega) = \nu(\Omega) = 10$. Prove that the class $\mathcal{C} := \{A \in \mathcal{F} : \mu(A) = \nu(A)\}$ is a monotone class. [4]

<u>*Question*</u> 3. Let $X, X_1, X_2, \cdots$ be real valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose $\lim_{n\to\infty} X_n = X$ in probability. Further suppose, almost surely $|X_n| \leq 5, \forall n$. Is $|X| \leq 5$ almost surely? Justify your answer. [4]

<u>*Question*</u> 4. Let $\{X_n\}$ be a sequence of real valued and independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, with each X_n having finite second moment. Let μ_n denote the Law of X_n, $m_n := \mathbb{E}X_n$, $S_n := X_1 + X_2 + \cdots + X_n, n \geq 1$ and $c_n^2 := Var(S_n)$. If

$$\lim_{n\to\infty} \frac{1}{c_n^{2.5}} \sum_{k=1}^{n} \mathbb{E}|X_k - m_k|^{2.5} = 0,$$

then for all $\epsilon > 0$, show that [5]

$$\lim_{n\to\infty} \frac{1}{c_n^2} \sum_{k=1}^{n} \int_{\{x\in\mathbb{R}:|x-m_k|\geq \epsilon c_n\}} (x - m_k)^2\, d\mu_k(x) = 0.$$

<u>*Question*</u> 5. Let X be a real valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}X^2 = \infty$. Is

$$\int_0^\infty \mathbb{P}(|X|^3 > x)\, dx$$

finite? Justify your answer. [6]

<u>*Question*</u> 6. For $\binom{x}{y} \in \mathbb{R}^2$, let $\|\binom{x}{y}\|_2 := \sqrt{x^2 + y^2}$ denote the standard Euclidean norm on $\mathbb{R}^2$. Let X and Y be two random vectors (defined on $(\Omega, \mathcal{F}, \mathbb{P})$) taking values in the square $[-1, 1] \times [-1, 1]$. Is $\mathbb{E}\|X\|_2^3$ finite? Justify your answer. Also prove that [2 + 4]

$$\left(\mathbb{E}\|X + Y\|_2^3\right)^{\frac{1}{3}} \leq \left(\mathbb{E}\|X\|_2^3\right)^{\frac{1}{3}} + \left(\mathbb{E}\|Y\|_2^3\right)^{\frac{1}{3}}.$$