MID SEMESTER EXAMINATION
PROBABILITY THEORY MTH309A
VENUE: L3 OROS
DATE AND TIME: FEBRUARY 21, 8:00 TO 10:00 HRS

General instructions:

- Each section below has specific instructions. Read them carefully.
- No notes or books are allowed during the exam.
- Maximum you can score: 35
- Notation: $\mathbb{B}_{\mathbb{R}^n}$ denotes the Borel σ-field on $\mathbb{R}^n$, generated by the open sets.

1. Section A

Question 1. Write down ALL correct choices in the following questions. You do not get any credit for the rough work. [1×5 = 5]

(i) For Borel subsets $A_1, A_2, \cdots, A_n$ of $\mathbb{R}$, define $A_1 + A_2 + \cdots + A_n := \{x_1 + x_2 + \cdots + x_n | x_i \in A_i, \forall i = 1, 2, \cdots, n\}$. Let Leb denote the Lebesgue measure on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$. Then

$$\lim_{n\to\infty} Leb(\underbrace{(-1,1] + (-1,1] + \cdots + (-1,1]}_{\text{n-times}})$$

(a) does not exist.
(b) is 0.
(c) is 2.
(d) is ∞.

(ii) Let μ_1 and μ_2 be probability measures on a measurable space $(\Omega, \mathcal{F})$. Identify probability measures in the following list. [1]
(a) $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ defined by $(\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2)(A) := \frac{1}{2}\mu_1(A) + \frac{1}{2}\mu_2(A), A \in \mathcal{F}$.
(b) $\frac{1}{2}\mu_1 + \frac{1}{2}$ defined by $(\frac{1}{2}\mu_1 + \frac{1}{2})(A) := \frac{1}{2}\mu_1(A) + \frac{1}{2}, A \in \mathcal{F}$.
(c) $2\mu_1 - \mu_2$ defined by $(2\mu_1 - \mu_2)(A) := 2\mu_1(A) - \mu_2(A), A \in \mathcal{F}$.

(iii) Let $f : (\mathbb{R}, \{\emptyset, \mathbb{R}\}) \to (\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ be a measurable function. Then [1]
(a) f is a non-constant differentiable function.
(b) f is continuous.
(c) f is constant.

(iv) Let F be the distribution function of a probability measure $\mathbb{P}$ on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$. Then the set $\{x \in \mathbb{R} \mid F \text{ is discontinuous at } x\}$ [1]
(a) need not be measurable.
(b) is a finite or countably infinite set.
(c) may be uncountable.
(d) is a Lebesgue null set.

(v) Let f and g be two real valued measurable functions defined on a measure space $(\Omega, \mathcal{F}, \mu)$. Assume that $f = g$ μ a.e.. Consider the function $h := f - g$. Then [1]
(a) $\int h \, d\mu$ need not exist.
(b) $\int h \, d\mu$ exists.
(c) $\int h \, d\mu = 0$.

2. Section B

Instructions: You need to answer only using the first principles (such as definitions). Marks will be deducted unless all the steps have been successfully explained. However, you are allowed to use the following results/facts without any further explanation, 1) generating sets of $\mathbb{B}_{\mathbb{R}^n}$, 2) $\mu(\emptyset) = 0$, if μ is a measure and 3) Monotone Convergence Theorem.

Question 2. We say that a subset A of $(0,1)$ is open if $A = B \cap (0,1)$, where B is some open set in $\mathbb{R}$. Let $\mathcal{C}$ denote the collection of open sets in $(0,1)$. Prove that the Borel σ-field on $(0,1)$, i.e. $\sigma(\mathcal{C})$, is the same as $\mathbb{B}_{\mathbb{R}} \cap (0,1) := \{B \cap (0,1) \mid B \in \mathbb{B}_{\mathbb{R}}\}$. [5]

Question 3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) := x + y$. Is f Borel measurable? Justify your answer. [5]

Question 4. Let X be an $\bar{\mathbb{R}}$ valued, integrable random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that X is finite almost surely (i.e. $\mathbb{P}$ a.e.). [5]

3. Section C

Instruction: You may use any result proved in class.

Question 5. (1) Let X be a real valued, bounded random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Is X integrable? Justify your answer. [2]

(2) Let X be a real valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Fix constants $c > 0, p > 0$. Prove the following variant of Markov inequality: [4]

$$\mathbb{P}(|X| \geq c) \leq \frac{1}{c^p}\mathbb{E}|X|^p.$$

(3) Let $\{X_n\}$ be a sequence of real valued random variables on the same probability space converging to X in the 4-th mean. Show that [2]

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \epsilon) = 0,$$

for any $\epsilon > 0$.

Question 6. (a) State the Dominated Convergence Theorem. [2]

(b) Compute the value of [6]

$$\lim_{n\to\infty} \int_1^3 \frac{n\sin(\frac{x}{n})}{x(1+x^2)}\, dx.$$

Question 7. (a) Let X be a real valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the following class of sets

$$\sigma(X) := \{X^{-1}(B) \mid B \in \mathbb{B}_{\mathbb{R}}\}.$$

Prove that $\sigma(X)$ is a σ-field. [2]

(b) Let X and Y be two real valued random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Is $\{X^{-1}(B) \mid B \in \mathbb{B}_{\mathbb{R}}\} \cup \{Y^{-1}(B) \mid B \in \mathbb{B}_{\mathbb{R}}\}$ a σ-field? Justify your answer. [2]

(c) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Consider the following class of sets

$$\mathcal{F}_\mu := \{A \cup N \mid A \in \mathcal{F}, N \subseteq B \text{ with } \mu(B) = 0, B \in \mathcal{F}\}.$$

Prove that $\mathcal{F}_\mu$ is a σ-field. [4]

4. Section D

Instructions: You may answer only one part of the following question. Only the first one will be graded if you answer both parts.

Question 8. (a) For any Borel set A in $\mathbb{R}$ and $x \in \mathbb{R}$, consider the set $A + x := \{y + x \mid y \in A\}$. We say that a measure μ on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ is translation invariant if

$$\mu(A + x) = \mu(A), \forall A \in \mathbb{B}_{\mathbb{R}}, x \in \mathbb{R}.$$

Show that there exists a positive constant c_μ, such that $\mu(A) = c_\mu Leb(A), \forall A \in \mathcal{F}$ for any translation invariant measure μ, where Leb denotes the Lebesgue measure on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$. [10]

(b) For $a, b \in (0,1]$, define

$$a \oplus_1 b := \begin{cases} a + b, \text{ if } a + b \leq 1 \\ a + b - 1, \text{ otherwise.} \end{cases}$$

Say that a is related to b, if there exists a rational number $r \in (0,1]$ such that $a \oplus_1 r = b$. Show that this is an equivalence relation. Let A be a set consisting of one representative from each equivalence class. Show that $A \notin \mathbb{B}_{\mathbb{R}}$. [4+6]