**QUIZ 2, MTH309A**
**TOTAL MARKS: 5**

ROLL NO:
NAME:

Instructions:

(1) Tick (✓) ALL correct answers among the options given. Illegible answers will be taken as incorrect. You get no credit for rough work. No extra pages will be supplied.
(2) You may refer to your own class notes. Searching in books/internet is not allowed.

Problems:

Q1. Fix $1 < p < \infty$. Then convergence in $p$-th mean implies [1]
  (a) convergence in $q$-th mean for any $q \in [1, p]$.
  (b) convergence in $q$-th mean for any $q \in [p, \infty)$.
  (c) almost sure convergence.
  (d) convergence in probability.

Q2. Let $X, Y$ and $Z$ be real valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the collection $\{X, Y, Z\}$ is independent if [1]
  (a) the joint distribution satisfies $F_{X,Y,Z}(x, y, z) = F_X(x)F_Y(y)F_Z(z), \forall x, y, z \in \mathbb{R}$.
  (b) all the random variables are absolutely continuous and the joint distribution has a density function given by $f_{X,Y,Z}(x, y, z) = f_X(x)f_Y(y)f_Z(z), \forall x, y, z \in \mathbb{R}$.
  (c) the collections $\{X, Y\}$, $\{Y, Z\}$ and $\{Z, X\}$ are independent.

Q3. Let $\{X_n\}$ be a sequence of real valued random variables on the same probability space such that the distribution of $X_n$ is $N(0, \frac{1}{n})$ for all $n$. Then [1]
  (a) the sequence $\{X_n\}$ converges to 0 in probability.
  (b) $\delta_0 \ll N(0, \frac{1}{n})$ for all $n$.
  (c) None of the above two statements can be determined from the given hypothesis.

Q4. Consider the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $2^{\mathbb{N}}$ is the power set of the set of natural numbers $\mathbb{N}$ and $\mu$ is the counting measure. Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ given by

$$f(m,n) := \begin{cases} m, & \text{if } m = n, \\ -m, & \text{if } n = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then [1]
  (a) $\int_{\mathbb{N}} \int_{\mathbb{N}} f(m,n)\, d\mu(m) d\mu(n) = \int_{\mathbb{N}} \int_{\mathbb{N}} f(m,n)\, d\mu(n) d\mu(m) = 0$.
  (b) $\int_{\mathbb{N}} \int_{\mathbb{N}} f(m,n)\, d\mu(m) d\mu(n) = \int_{\mathbb{N}} \int_{\mathbb{N}} f(m,n)\, d\mu(n) d\mu(m) = \infty$.
  (c) $\int_{\mathbb{N}} \int_{\mathbb{N}} f(m,n)\, d\mu(m) d\mu(n) = 0, \int_{\mathbb{N}} \int_{\mathbb{N}} f(m,n)\, d\mu(n) d\mu(m) = \infty$.
  (d) $\int_{\mathbb{N}} \int_{\mathbb{N}} f(m,n)\, d\mu(m) d\mu(n) = \infty, \int_{\mathbb{N}} \int_{\mathbb{N}} f(m,n)\, d\mu(n) d\mu(m) = 0$.

Q5. Consider the Lebesgue measure *Leb* restricted to the interval $(0, 1]$. Let $A_n(x)$ denote the number of occurences of the digit 1 in the first $n$ digits of (non-terminating) decimal expansion of the number $x$. Suppose that[1] for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} Leb\left(x \in (0,1] : \left|\frac{A_n(x)}{n} - \frac{1}{10}\right| > \epsilon\right) < \infty.$$

Then [1]
  (a) *Leb* is a probability measure on $((0, 1], \mathbb{B}_{(0,1]})$.
  (b) $Leb\left(x \in (0,1] : \left|\frac{A_n(x)}{n} - \frac{1}{10}\right| > \epsilon \text{ infinitely often}\right) = 0$ for all $\epsilon > 0$.
  (c) $Leb\left(x \in (0,1] : \left|\frac{A_n(x)}{n} - \frac{1}{10}\right| > \epsilon \text{ infinitely often}\right) = 1$ for all $\epsilon > 0$.

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*Date*: March 11, 2019. Time: 14:00 - 14:50 hrs.

[1]this fact is related to Borel's Normal Number theorem. Take a look after the quiz if you are interested.