MID SEMESTER EXAMINATION (2019-20 EVEN),
PROBABILITY THEORY (MTH309A)
TOTAL MARKS: 32, MAXIMUM YOU CAN SCORE: 30

ROLL NO:

NAME:

Instructions:

(1) Illegible answers will be taken as incorrect.

(2) You may refer to your own class notes and use results proved in class. Searching in books/internet is not allowed. Mobiles must be switched off during the examination.

(3) At the end of the exam, return the question paper along with your answer-script.

(4) $(\Omega, \mathcal{F})$ denotes some arbitrary but fixed measurable space.

1. Section A

Question 1. This question consists of Multiple Select Questions. The number of correct option(s) can be one or more than one. Put a tick (✓) beside only the correct options to get credit. Do not write in the answer-script. Make sure to return the question paper along with your answer-script at the end of the exam. You get no credit for rough work. No partial credits will be awarded.

(I) The statement 'the set of irrational numbers is a Borel subset of $\mathbb{R}$' is [2]

✓(a) True. (b) False.

(II) Given $A_1, A_2, \cdots \in \mathcal{F}$, which of the following sets are also in $\mathcal{F}$? [2]

✓(a) $A_1 \triangle A_2 := (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$. ✓$(b)$ $A_1 \setminus A_2$. (c) $A_1 \times A_2$. ✓(d) $\limsup_{n\to\infty} A_n$.

(III) Let $f, g : (\mathbb{R}, \mathbb{B}_{\mathbb{R}}) \to (\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ be two simple functions. Consider the function $h := \max\{f, g\}$. Then [2]

(a) for all simple functions f, there exists a simple function g, such that h is not a simple function.

✓(b) h must be a simple function for all such f and g.

(c) h is not a simple function for all such f and g.

(d) none of the above statements are true.

(IV) Let $\mathbb{P}$ be a probability measure on $\mathcal{F}$ and let $\{A_n\}$ be a sequence of sets in $\mathcal{F}$ such that $\mathbb{P}(A_n) = 1, \forall n$. Then $\mathbb{P}(\bigcap_{n=1}^{\infty} A_n)$ equals [2]

(a) 0. (b) $\frac{1}{2}$. ✓(c) 1. (d) can be any number in $(0, 1)$.

(V) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := \exp\left(-\frac{|x|}{2}\right), x \in \mathbb{R}$. Then the statement 'the function f can be uniformly approximated by simple functions' is [2]

✓(a) True. (b) False.

Date: February 20, 2020. Time: 8:00 - 10:00 hrs. Venue: L10-L11 OROS.

(VI) Consider the function $f(x) = x^3, x \in \mathbb{R}$ and the measure $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{3}\delta_0 + \frac{1}{6}\delta_1$ on $(\mathbb{R}, \mathbb{B}_\mathbb{R})$. Then [2]

✓(a) f is integrable with respect to the measure μ.

(b) $\int_\mathbb{R} |f(x)|\, d\mu(x) = \infty$.

✓(c) $\int_\mathbb{R} f(x)\, d\mu(x) = -\frac{1}{3}$.

(d) $\int_\mathbb{R} |f(x)|\, d\mu(x) = 1$.

2. Section B

Instructions: Answer these questions in your answer-script and write answers to sub-parts consecutively. Justify your answers.

Question 2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Fix $A \in \mathcal{F}$ with $A \neq \emptyset$. Consider the class of sets

$$\mathcal{D} := \{B \in \mathcal{F} : \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)\}.$$

Is $\mathcal{D}$ non-empty? Is it closed under complimentation? Is it a Monotone class? [1 + 1 + 3]

Hints: Since $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$, we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for $B = \emptyset, \Omega$. Hence $\phi, \Omega \in \mathcal{D}$ and $\mathcal{D}$ is non-empty.

If $B \in \mathcal{D}$, then $\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = \mathbb{P}(A)[1 - \mathbb{P}(B)] = \mathbb{P}(A)\mathbb{P}(B^c)$. Hence $B^c \in \mathcal{D}$ and hence $\mathcal{D}$ is closed under complimentation.

If $\{B_n\}$ be an increasing sequence of sets in $\mathcal{D}$, then using the continuity from below of $\mathbb{P}$, we have

$$\mathbb{P}(A \bigcap \left(\bigcup_n B_n\right)) = \lim_n \mathbb{P}(A \cap B_n) = \mathbb{P}(A) \lim_n \mathbb{P}(B_n) = \mathbb{P}(A)\mathbb{P}\left(\bigcup_n B_n\right).$$

Hence $\bigcup_n B_n \in \mathcal{D}$ and $\mathcal{D}$ is closed under countable increasing unions.

To show closure under countable decreasing intersections, either use continuity from above of $\mathbb{P}$ or use a combination of closure under countable increasing unions and complimentation.

Hence $\mathcal{D}$ is a Monotone class.

Question 3. (a) Let μ and ν be two probability measures on $(\mathbb{R}, \mathbb{B}_\mathbb{R})$. Let F_μ and F_ν denote the corresponding distribution functions. If $F_\mu(x) = F_\nu(x), \forall x \in \mathbb{R}$, it is necessary that $\mu = \nu$? [3]

(b) Let X be a real valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Fix $x \in \mathbb{R}$ and define the random variable $Y(\omega) := 1_{(X \leq x)}(\omega), \forall \omega \in \Omega$. Find the law of Y. [2]

Hints:(a) (Proved in class) Recall that $\mathcal{D} := \{A \in \mathbb{B}_\mathbb{R} : \mu(A) = \nu(A)\}$ is a Monotone class.

Hypothesis implies $(-\infty, x] \in \mathcal{D}, \forall x \in \mathbb{R}$. Check that $(a, b] \in \mathcal{D}$ for all $a, b \in \mathbb{R}, a < b$. Check that finite disjoint unions of $(a, b]$ are also in $\mathcal{D}$. But finite disjoint unions of $(a, b]$ is a field and generates $\mathbb{B}_\mathbb{R}$. By the Monotone class Theorem $\mathcal{D} = \mathbb{B}_\mathbb{R}$ and hence $\mu = \nu$.

(b) Consider the set $A := (X \leq x) = \{\omega \in \Omega : X(\omega) \leq x\}$. Since X is a random variable, $A \in \mathcal{F}$. Moreover, $Y = 1_A$.

Then for any $B \in \mathbb{B}_{\mathbb{R}}$,

$$Y^{-1}(B) = \begin{cases} \emptyset, \text{ if } 0, 1 \notin B, \\ A, \text{ if } 0 \notin B, 1 \in B, \\ A^c, \text{ if } 0 \in B, 1 \notin B, \\ \Omega, \text{ if } 0, 1 \in B. \end{cases}$$

Now compute $\mathbb{P} \circ Y^{-1}$.

Question 4. Let $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ be a measurable function and let μ be a σ-finite measure on $(\Omega, \mathcal{F})$. Recall that f^+ and f^- denote the positive and negative parts of f respectively.

(a) Define $\nu^+(A) := \int_A f^+ \, d\mu, \forall A \in \mathcal{F}$ and $\nu^-(A) := \int_A f^- \, d\mu, \forall A \in \mathcal{F}$. Are ν^+ and ν^- σ-finite measures on $(\Omega, \mathcal{F})$? [2]

(b) Suppose f is integrable with respect to the measure μ. Are ν^+ and ν^- finite measures on $(\Omega, \mathcal{F})$? [1]

(c) Suppose f is integrable with respect to the measure μ. Consider the set function $\nu(A) := \nu^+(A) - \nu^-(A), \forall A \in \mathcal{F}$. Is the domain of ν the σ-field $\mathcal{F}$? (i.e., is ν defined for all sets $A \in \mathcal{F}$?). Is ν countably additive on its domain, i.e. given a sequence of pairwise disjoint sets $\{A_n\}$ from its domain, is $\nu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n)$? [0.5 + 1.5]

Hints:(a) (Proved in class) Since f^+ and f^- are non-negative measurable functions, ν^+ and ν^- are measures. If μ is σ-finite, let $\{\Omega_n\}$ be a sequence in $\mathcal{F}$ such that $\mu(\Omega_n) < \infty, \forall n$ and $\Omega_n \uparrow \Omega$.

Consider the sets $(f^+ \leq n) \cap \Omega_n$ and $(f^- \leq n) \cap \Omega_n$ and prove the σ-finiteness of ν^+ and ν^-.

(b) Given that $\nu^+(\Omega) = \int_\Omega f^+ \, d\mu < \infty$ and $\nu^-(\Omega) = \int_\Omega f^- \, d\mu < \infty$, since f is integrable with respect to the measure μ. Hence ν^+ and ν^- are finite measures.

(c) By part (b), $\nu^+(A) < \infty$ and $\nu^-(A) < \infty$ for all $A \in \mathcal{F}$. Hence, ν is defined for all $A \in \mathcal{F}$.

Note that $\sum_n |\nu(A_n)| \leq \sum_n \nu^+(A_n) + \sum_n \nu^-(A_n) \leq \nu^+(\Omega) + \nu^-(\Omega) < \infty$, i.e. the series $\sum_n \nu(A_n)$ is absolutely convergent. Therefore,

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \nu^+\left(\bigcup_{n=1}^{\infty} A_n\right) - \nu^-\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu^+(A_n) - \sum_{n=1}^{\infty} \nu^-(A_n) = \sum_{n=1}^{\infty} \nu(A_n).$$

Question 5. Let $X_1, X_2, \cdots$ be a sequence of real valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) Suppose $\lim_{n\to\infty} X_n(\omega)$ exists for all $\omega \in \Omega$. Is the function $X(\omega) := \lim_{n\to\infty} X_n(\omega), \forall \omega \in \Omega$ a random variable on the space $(\Omega, \mathcal{F}, \mathbb{P})$? [1]

(b) Suppose that $|X_n(\omega)| \leq$ (your roll number), $\forall n, \omega$. Is X, as defined in part (a), integrable with respect to the measure $\mathbb{P}$? What can you say about $\lim_{n\to\infty} \mathbb{E}((X_n - X)^3)$? [1 + 3]

Hints:(a) (Proved in class) Limit of a sequence of measurable functions is measurable. Here X is the limit of measurable functions on a probability space and hence is a random variable.

(b) Suppose M be your roll number. Check that $|X(\omega)| \leq M, \forall \omega$. Then $\int_\Omega |X(\omega)| \, d\mathbb{P}(\omega) \leq \int_\Omega M \, d\mathbb{P}(\omega) \leq M\mathbb{P}(\Omega) = M < \infty$. Hence X is integrable.

Consider the random variables $Y_n := (X_n - X)^3$. Then the sequence $\{Y_n\}$ converges to 0 pointwise and $|Y_n| \leq |X_n - X|^3 \leq (2M)^3, \forall n$, i.e. uniformly bounded. Check that DCT is applicable for the sequence.

Apply DCT to conclude $\lim_{n\to\infty} \mathbb{E}((X_n - X)^3) = \lim_{n\to\infty} \mathbb{E} Y_n = \mathbb{E} \lim_{n\to\infty} Y_n = 0$.