MID SEMESTER EXAMINATION (2019-20 EVEN),
PROBABILITY THEORY (MTH309A)
TOTAL MARKS: 32, MAXIMUM YOU CAN SCORE: 30

ROLL NO:

NAME:

Instructions:

(1) Illegible answers will be taken as incorrect.

(2) You may refer to your own class notes and use results proved in class. Searching in books/internet is not allowed. Mobiles must be switched off during the examination.

(3) At the end of the exam, return the question paper along with your answer-script.

(4) $(\Omega, \mathcal{F})$ denotes some arbitrary but fixed measurable space.

1. Section A

Question 1. This question consists of Multiple Select Questions. The number of correct option(s) can be one or more than one. Put a tick (✓) beside only the correct options to get credit. Do not write in the answer-script. Make sure to return the question paper along with your answer-script at the end of the exam. You get no credit for rough work. No partial credits will be awarded.

(I) The statement 'the set of irrational numbers is a Borel subset of $\mathbb{R}$' is [2]

(a) True. (b) False.

(II) Given $A_1, A_2, \cdots \in \mathcal{F}$, which of the following sets are also in $\mathcal{F}$? [2]

(a) $A_1 \triangle A_2 := (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$. (b) $A_1 \setminus A_2$. (c) $A_1 \times A_2$. (d) $\limsup_{n \to \infty} A_n$.

(III) Let $f, g : (\mathbb{R}, \mathbb{B}_{\mathbb{R}}) \to (\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ be two simple functions. Consider the function $h := \max\{f, g\}$. Then [2]

(a) for all simple functions f, there exists a simple function g, such that h is not a simple function.
(b) h must be a simple function for all such f and g.
(c) h is not a simple function for all such f and g.
(d) none of the above statements are true.

(IV) Let $\mathbb{P}$ be a probability measure on $\mathcal{F}$ and let $\{A_n\}$ be a sequence of sets in $\mathcal{F}$ such that $\mathbb{P}(A_n) = 1, \forall n$. Then $\mathbb{P}(\bigcap_{n=1}^{\infty} A_n)$ equals [2]

(a) 0. (b) $\frac{1}{2}$. (c) 1. (d) can be any number in $(0, 1)$.

(V) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := \exp\left(-\frac{|x|}{2}\right), x \in \mathbb{R}$. Then the statement 'the function f can be uniformly approximated by simple functions' is [2]

(a) True. (b) False.

Date: February 20, 2020. Time: 8:00 - 10:00 hrs. Venue: L10-L11 OROS.

(VI) Consider the function $f(x) = x^3, x \in \mathbb{R}$ and the measure $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{3}\delta_0 + \frac{1}{6}\delta_1$ on $(\mathbb{R}, \mathbb{B}_\mathbb{R})$. Then [2]

(a) f is integrable with respect to the measure μ.

(b) $\int_\mathbb{R} |f(x)|\, d\mu(x) = \infty$.

(c) $\int_\mathbb{R} f(x)\, d\mu(x) = -\frac{1}{3}$.

(d) $\int_\mathbb{R} |f(x)|\, d\mu(x) = 1$.

2. Section B

Instructions: Answer these questions in your answer-script and write answers to sub-parts consecutively. Justify your answers.

Question 2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Fix $A \in \mathcal{F}$ with $A \neq \emptyset$. Consider the class of sets

$$\mathcal{D} := \{B \in \mathcal{F} : \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)\}.$$

Is $\mathcal{D}$ non-empty? Is it closed under complimentation? Is it a Monotone class? [1 + 1 + 3]

Question 3. (a) Let μ and ν be two probability measures on $(\mathbb{R}, \mathbb{B}_\mathbb{R})$. Let F_μ and F_ν denote the corresponding distribution functions. If $F_\mu(x) = F_\nu(x), \forall x \in \mathbb{R}$, it is necessary that $\mu = \nu$? [3]

(b) Let X be a real valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Fix $x \in \mathbb{R}$ and define the random variable $Y(\omega) := 1_{(X \leq x)}(\omega), \forall \omega \in \Omega$. Find the law of Y. [2]

Question 4. Let $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathbb{B}_\mathbb{R})$ be a measurable function and let μ be a σ-finite measure on $(\Omega, \mathcal{F})$. Recall that f^+ and f^- denote the positive and negative parts of f respectively.

(a) Define $\nu^+(A) := \int_A f^+\, d\mu, \forall A \in \mathcal{F}$ and $\nu^-(A) := \int_A f^-\, d\mu, \forall A \in \mathcal{F}$. Are ν^+ and ν^- σ-finite measures on $(\Omega, \mathcal{F})$? [2]

(b) Suppose f is integrable with respect to the measure μ. Are ν^+ and ν^- finite measures on $(\Omega, \mathcal{F})$? [1]

(c) Suppose f is integrable with respect to the measure μ. Consider the set function $\nu(A) := \nu^+(A) - \nu^-(A), \forall A \in \mathcal{F}$. Is the domain of ν the σ-field $\mathcal{F}$? (i.e., is ν defined for all sets $A \in \mathcal{F}$?). Is ν countably additive on its domain, i.e. given a sequence of pairwise disjoint sets $\{A_n\}$ from its domain, is $\nu\left(\cup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \nu(A_n)$? [0.5 + 1.5]

Question 5. Let $X_1, X_2, \cdots$ be a sequence of real valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) Suppose $\lim_{n\to\infty} X_n(\omega)$ exists for all $\omega \in \Omega$. Is the function $X(\omega) := \lim_{n\to\infty} X_n(\omega), \forall \omega \in \Omega$ a random variable on the space $(\Omega, \mathcal{F}, \mathbb{P})$? [1]

(b) Suppose that $|X_n(\omega)| \leq$ (your roll number)$, \forall n, \omega$. Is X, as defined in part (a), integrable with respect to the measure $\mathbb{P}$? What can you say about $\lim_{n\to\infty} \mathbb{E}((X_n - X)^3)$? [1 + 3]