MTH309A PRACTICE PROBLEMS SET 1

AshBk = Probability & Measure Theory (2nd Edition), Robert B. Ash and Catherine A. Doléans-Dade. Elsevier.

ChungBk = A Course in Probability Theory (3rd Edition), Kai Lai Chung. Academic Press (Elsevier).

Question 1. AshBk [pp. 10-12] Problems 1, 3, 4, 5, 8, 9, 12

Question 2. ChungBk [pp. 19-21] Exercises 4, 12 [Terminology: B.F. = Borel Field = σ-field]

Question 3. In class, we have listed collections of different generating sets for the Borel σ-field on $\mathbb{R}$ and have verified for some of the collections that they do generate the same σ-field. Verify for the rest in the list for which the verification was not done in class.

Question 4. Suppose that a collection $\mathcal{F}$ of subsets of Ω has the following properties.

(1) $\emptyset \in \mathcal{F}$.
(2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
(3) If $A \in \mathcal{F}$, then A^c is a finite disjoint union of members of $\mathcal{F}$.

Consider another collection $\mathcal{C}$ consisting of finite disjoint union of members of $\mathcal{F}$. Show that $\mathcal{C}$ is a field of subsets of Ω.

Remarks: (not included in the exercise) The collection of left-open right-closed intervals in $\mathbb{R}$ is an example of such an $\mathcal{F}$. Recall that we consider intervals of the form $(a, \infty), a \in \mathbb{R}$ to be left-open right-closed. This collection $\mathcal{C}$ was mentioned in the class as an example of a field, but not a σ-field.

Question 5. Let $f : \Omega \to \Omega'$ be a function and let $\mathcal{F}$ be a σ-field in Ω'. Show that $f^{-1}(\mathcal{F}) := \{f^{-1}(A) : A \in \mathcal{F}\}$ is a σ-field in Ω.

Question 6. Let $f : \Omega \to \Omega'$ be a function and let $\mathcal{F}$ be a collection of subsets of Ω'. Show that

$$\sigma(f^{-1}(\mathcal{F})) = f^{-1}(\sigma(\mathcal{F})),$$

where $f^{-1}(A)$ denotes the pre-image of $A \subset \Omega'$, $f^{-1}(\mathcal{F}) := \{f^{-1}(A) : A \in \mathcal{F}\}$ and $f^{-1}(\sigma(\mathcal{F})) := \{f^{-1}(A) : A \in \sigma(\mathcal{F})\}$. (Hint: Use the principle of good sets.)

Question 7. Say that a σ-field is *countably generated (or separable)* if it can be generated by some countable collection of sets. Show that the Borel σ-field $\mathcal{B}_{\mathbb{R}}$ on $\mathbb{R}$ is countably generated.

Question 8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. Fix $A \in \mathcal{F}$. Show that

$$\mathcal{C} := \{B \in \mathcal{F} : \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)\}$$

is a Monotone class.

Question 9. Suppose that $\mathcal{F}$ is a collection of subsets of Ω with the following properties: $\Omega \in \mathcal{F}$ and $A, B \in \mathcal{F}$ implies $A \setminus B := A \cap B^c \in \mathcal{F}$. Show that $\mathcal{F}$ is a field.

Question 10. Find examples of Monotone classes which are not σ-fields.