# MTH309A PRACTICE PROBLEMS SET 5

AshBk = Probability & Measure Theory (2nd Edition), Robert B. Ash and Catherine A. Doléans-Dade. Elsevier.

ChungBk = A Course in Probability Theory (3rd Edition), Kai Lai Chung. Academic Press (Elsevier).

*Question* 1. AshBk [pp. 54-55] Problems 2, 4-7; [pp. 58-59] Problems 1, 4; [pp. 94-95] Problems 1-2, 4-6

*Question* 2. ChungBk [pp. 51] Exercises 12-14

*Question* 3. Let $\lambda$ denote the Lebesgue measure on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$. For any Borel set $A$ and $x \in \mathbb{R}$, define $A + x := \{a + x : a \in A\}$. Show that $\lambda$ is translation invariant, i.e.

$$\lambda(A) = \lambda(A + x), \, \forall x \in \mathbb{R}, A \in \mathbb{B}_{\mathbb{R}}.$$

Note: Also check Q4, set 4.

*Question* 4. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable and fix $a \in \mathbb{R}$. Consider the two integrals '$\int_{\mathbb{R}} f(x)\, dx$' and '$\int_{\mathbb{R}} f(x-a)\, dx$' with respect to the Lebesgue measure. If one integral exists, show the existence of the other. In this case, show that the two integrals are actually equal.

*Question* 5. Let $f, g$ be integrable functions on $(\Omega, \mathcal{F}, \mu)$ such that

$$\int_A f \, d\mu \leq \int_A g \, d\mu, \ \forall A \in \mathcal{F}.$$

Show that $f \leq g$, $\mu$-a.e..

*Question* 6. Fix $a, b \in \mathbb{R}$ with $a < b$. Suppose that there exist functions $f : (a, b) \times \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ such that

(1) $\omega \in \Omega \mapsto f(t, \omega)$ is $\mu$-integrable for every fixed $t \in (a, b)$,
(2) $t \in (a, b) \mapsto f(t, \omega)$ is continuous for every fixed $\omega \in \Omega$,
(3) $g \geq 0$ and $\mu$-integrable,
(4) $|f(t, \omega)| \leq g(\omega), \forall t, \omega$.

Then show that the function $h : (a, b) \to \mathbb{R}$ defined by

$$h(t) := \int_\Omega f(t, \omega)\, d\mu(\omega)$$

is continuous.

*Question* 7. Fix $a, b \in \mathbb{R}$ with $a < b$. Suppose that there exist functions $f : (a, b) \times \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ such that

(1) $\omega \in \Omega \mapsto f(t, \omega)$ is $\mu$-integrable for every fixed $t \in (a, b)$,
(2) $t \in (a, b) \mapsto f(t, \omega)$ is differentiable (in $t$) for every fixed $\omega \in \Omega$,
(3) $g \geq 0$ and $\mu$-integrable,
(4) $|\frac{\partial}{\partial t} f(t, \omega)| \leq g(\omega), \forall t, \omega$.

Then show that the function $h : (a, b) \to \mathbb{R}$ defined by

$$h(t) := \int_\Omega f(t, \omega)\, d\mu(\omega)$$

is differentiable with $h'(t) = \int_\Omega \frac{\partial}{\partial t} f(t, \omega)\, d\mu(\omega)$.

*Question* 8. (Hölder's inequality) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Fix $p, q \in (1, \infty)$ with $p^{-1} + q^{-1} = 1$. Then for any $f \in \mathcal{L}^p(\mu), g \in \mathcal{L}^q(\mu)$, we have $fg \in \mathcal{L}^1(\mu)$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$