ANSWERS TO QUIZ 2, MTH309A

Instructions:

(1) Illegible answers will be taken as incorrect.

(2) You get no credit for rough work. No extra pages will be supplied.

(3) You may refer to your own class notes. Searching in books/internet is not allowed. Mobiles must be switched off during the quiz.

(4) $(\Omega, \mathcal{F}, \mathbb{P})$ stands for an arbitrary probability space.

Multiple Choice Questions [Only one option is correct. Put a tick (✓) beside the correct option.]

Question 1. Let $\{A_n\}$ be a sequence in $\mathcal{F}$ with $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. Then the value of $\mathbb{P}(\liminf_n A_n)$ [1]

(a) is 1.

✓(b) is 0.

(c) is $\frac{1}{2}$.

(d) can not be determined from the given information.

Question 2. Let $A := [0,1] \cap \mathbb{Q}$, where $\mathbb{Q}$ denotes the set of rationals in $\mathbb{R}$. Consider the function $f : [0,1] \to \mathbb{R}$ defined as $f(x) := 1_A(x)$. Then f [1]

(a) is Riemann integrable as well as Lebesgue integrable.

(b) is Riemann integrable, but not Lebesgue integrable.

✓(c) is Lebesgue integrable, but not Riemann integrable.

(d) is neither Riemann integrable, nor Lebesgue integrable.

Question 3. Let μ be a measure on $(\Omega, \mathcal{F})$ and let $f \in \mathcal{L}^2(\mu)$. Then the set function $A \in \mathcal{F} \mapsto \int_A f^2 d\mu$ [1]

(1) is countably additive, but not necessarily non-negative.

(2) is non-negative, but not necessarily countably additive.

✓(3) is a finite measure.

(4) is an infinite measure.

Question 4. Let μ be a probability measure on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$. Fix $t \in \mathbb{R}$. The function $x \in \mathbb{R} \mapsto e^{itx} \in \mathbb{C}$ is in $\mathcal{L}^p(\mu; \mathbb{C})$ [1]

(a) only for $p = 1$.

(b) only for $p = 2$.

✓(c) for $p \in [1, \infty)$.

(d) for no values of $p \in (0, \infty)$.

Date: March 4, 2020. Time: 11:00 - 11:50 hrs.

Multiple Select Questions [Number of correct option(s) can be one or more than one. Put a tick (✓) beside only the correct option(s) to get credit. No partial credits will be awarded.]

Question 5. Let X be a real valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Which of the following condition(s) implies that $X \in \mathcal{L}^1(\mathbb{P})$? [2]

✓(a) $\int_\Omega X^+ \, d\mathbb{P} < \infty$ and $\int_\Omega X^- \, d\mathbb{P} < \infty$.

✓(b) There exists a constant $C > 0$ such that $\mathbb{P}(\{\omega : |X(\omega)| > C\}) = 0$.

✓(c) $\mathbb{E}|X| < \infty$.

✓(d) There exists a random variable Y such that $|X| \leq Y$ a.s. and $Y \in \mathcal{L}^1(\mathbb{P})$.

Question 6. Let $\{f_n\}$ be a sequence of simple functions on $\mathbb{R}$ given by $f_n(x) := (2n+1)1_{A_n}(x), \forall x \in \mathbb{R}, n = 1, 2, \cdots$ where $A_n = (0, \frac{1}{n}]$. Let Leb denote the Lebesgue measure on $\mathbb{R}$. Then [2]

(a) $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) \, dLeb(x)$ does not exist, but $\int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x) \, dLeb(x) = 0$.

(b) $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) \, dLeb(x) = 0$, but $\int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x) \, dLeb(x)$ does not exist.

(c) $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) \, dLeb(x) = \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x) \, dLeb(x) = 0$.

✓(d) $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) \, dLeb(x) = 2$, but $\int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x) \, dLeb(x) = 0$..

Question 7. Let f, g be two real valued Borel measurable functions on a measure space $(\Omega, \mathcal{F}, \mu)$. Let $(\Omega, \mathcal{F}_\mu, \bar{\mu})$ denote the completion of the given measure space. Identify the correct option(s). [2]

✓(a) If $f = g$ μ-a.e. and f is integrable, then g is integrable.

✓(b) If $f = 0$ μ-a.e., then $f = 0$ $\bar{\mu}$-a.e..

✓(c) f is $\mathcal{F}_\mu/\mathbb{B}_\mathbb{R}$ measurable.

✓(d) If $\int_\Omega f \, d\mu$ exists, then so does $\int_\Omega f \, d\bar{\mu}$ and $\int_\Omega f \, d\mu = \int_\Omega f \, d\bar{\mu}$.