END SEMESTER EXAMINATION (2019-20 ODD),
BASIC PROBABILITY AND DISTRIBUTION THEORY (MTH431A)
TOTAL MARKS: 46, MAXIMUM YOU CAN SCORE: 40

ROLL NO:

NAME:

Instructions:

(1) Illegible answers will be taken as incorrect.

(2) You may refer to your own class notes and use results proved in class. Searching in books/internet is not allowed. Mobiles must be switched off during the examination.

(3) At the end of the exam, return the question paper along with your answer-script.

(4) $\mathbb{N} = \{1, 2, \cdots\}$ denotes the set of natural numbers. $(\Omega, \mathcal{F}, \mathbb{P})$ denotes some arbitrary but fixed probability space. Unless stated otherwise, all random variables and random vectors are assumed to be defined on this probability space.

1. Section A

Question 1. This question consists of MCQs with exactly one correct answer in each. Put a tick (✓) beside the correct option. Do not write your answer in the answer-script. You get no credit for rough work.

(I) For any real valued random variable X with $\mathbb{E}|X|^3 < \infty$, define the *coefficient of skewness* as

$$\frac{\mathbb{E}\left[(X - \mathbb{E}X)^3\right]}{(\mathbb{E}[(X - \mathbb{E}X)^2])^{\frac{3}{2}}}.$$

The coefficient of skewness of $Poisson(4)$ is given by [3]

(a) 2^{-1}. (b) 4^{-1}. (c) 8^{-1}. (d) 16.

(II) Let X be a real valued random variable with $\mathbb{E}X^2 < \infty$. Then the series '$\sum_{n=1}^{\infty} \mathbb{P}(|X| > \sqrt{n})$' [2]

(a) converges. (b) diverges to ∞. (c) may converge or diverge for different X.

(III) For any $A \in \mathcal{F}$, consider the random variable

$$1_A(\omega) := \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{otherwise.} \end{cases}$$

For arbitrary $A, B \in \mathcal{F}$, consider the statement '$\mathbb{E}\ [1_A \mid 1_B = 1] = \mathbb{P}(A \mid B)$'. This statement is [2]

(a) false for all $A, B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$.

Date: November 18, 2019. Time: 9:00 - 12:00 hrs. Venue: L2 ERES.

(b) true for all $A, B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$.

(c) true only if $A = B = \Omega$.

(IV) Let $X \sim N(\mu, \sigma^2)$ with $\mathbb{E}X^3 = 0$. Then [3]

(a) $\mu = 0$.

(b) there are many possible values of μ, including 0.

(c) there are many possible values of μ. However $\mu \neq 0$.

(d) there is not enough information to conclude about values of μ.

(V) Let $X_1, X_2, \cdots$ be a sequence of independent Cauchy random variables with p.d.f. $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \forall x \in \mathbb{R}$. Define $Y_1 := X_1$ and $Y_n := \frac{1}{n}(X_1 + \cdots + X_n)$, for $n > 1$. Then the statement '$Y_1, Y_2, \cdots$ are all identically distributed' is [2]

(a) true. (b) false.

(VI) Fix $p_1, p_2, \cdots, p_k > 0$ with $\sum_i p_i = 1$. Let the random vector $(X_1, X_2, \cdots, X_{k-1})^t$ follow a multinomial distribution with joint p.m.f. given by

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \cdots, X_{k-1} = x_{k-1}) = \frac{n!}{x_1!x_2!\cdots x_{k-1}!(n - \sum_{i=1}^{k-1} x_i)!} p_1^{x_1} p_2^{x_2} \cdots p_{k-1}^{x_{k-1}} p_k^{n - \sum_{i=1}^{k-1} x_i},$$

if $x_1 + x_2 + \cdots + x_{k-1} \leq n$ and 0 otherwise. For $1 \leq i, j \leq k-1$ and $i \neq j$, the Correlation coefficient between X_i and X_j is [3]

(a) 0 (b) 1 (c) $\left[\frac{p_i p_j}{(1-p_i)(1-p_j)}\right]^{\frac{1}{2}}$ (d) $-\left[\frac{p_i p_j}{(1-p_i)(1-p_j)}\right]^{\frac{1}{2}}$.

Question 2. This question consists of MSQs with the number of correct options being 1, 2, 3 or 4. Put a tick (✓) beside only the correct option(s) to get credit. Do not write your answer in the answer-script. No partial credits will be awarded. You get no credit for rough work.

(I) Fix $A \in \mathcal{F}$. Consider the collection of sets $\mathcal{C} := \{B \in \mathcal{F} : \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)\}$. Then [4]

(a) $\mathcal{C}$ is non-empty.

(b) $\mathcal{C}$ is closed under complementation.

(c) $\mathcal{C}$ is closed under countable increasing unions.

(d) $\mathcal{C}$ is closed under countable intersections.

(II) Let X and Y be independent discrete random variables. Further assume $X \sim Binomial(n, p)$ for some integer $n \geq 1$ and $p \in (0, 1)$. Define $Z := 1 + X$. Which of the following are also random variables? [3]

(a) YZ (b) $\frac{Y}{Z}$ (c) $\sqrt{YZ}$ (d) $\min\{Y, Z\}$.

(III) Let X, Y be exchangable random variables. Then [2]

(a) X and Y are independent and identically distributed.

(b) X and Y are identically distributed, but not necessarily independent.

(c) $X - Y$ and $Y - X$ are identically distributed.

(d) none of the above.

(IV) Let $X = (X_1, X_2, X_3)^t$ be a continuous random vector taking values in $\mathbb{R}^3$. Then the distribution of X is determined by [2]

(a) the joint p.d.f. of X.

(b) the Characteristic function of X.

(c) the distribution function of X.

(d) all 2-dimensional marginal distributions, if the components X_1, X_2, X_3 are independent.

2. Section B

Instructions: Answer these questions in your answer-script. You can use any result proved in class.

Question 3. Let X_1, X_2 and X_3 be independent Uniform$(0,1)$ random variables. Let $X_{(1)}, X_{(2)}$ and $X_{(3)}$ denote the corresponding order statistics. Identify the distribution of $X_{(1)}$ and compute $\mathbb{E}X_{(1)}$. Justify your answer. [4 + 1]

Question 4. Let X be a continuous random variable with a continuous pdf f and distribution function F. Suppose that there exists $\alpha \in \mathbb{R}$ such that

$$\mathbb{P}(X \geq \alpha + x) = \mathbb{P}(X \leq \alpha - x), \forall x \in \mathbb{R}.$$

(a) Identify the functions $F(\alpha + x) + F(\alpha - x), x \in \mathbb{R}$ and $f(\alpha + x) - f(\alpha - x), x \in \mathbb{R}$. Justify your answer. [0.5 + 0.5]

(b) What can you say about the mean and median of X? Justify your answer. [2 + 2]

Question 5. (a) Using Hölder's inequality or otherwise, prove the following inequality

$$(\mathbb{E}|X+Y|^p)^{\frac{1}{p}} \leq (\mathbb{E}|X|^p)^{\frac{1}{p}} + (\mathbb{E}|Y|^p)^{\frac{1}{p}},$$

for real valued continuous random variables X, Y and any real number $p > 1$. [3]

(b) Let X and Y be two random variables with Exponential(1) distribution, such that the Correlation coefficient between X and Y is $-\frac{1}{3}$. Is this information sufficient to compute the value of $\mathbb{E}\left[\mathbb{E}[X^2 \mid Y]\right]$? If your answer is yes, then compute the value. If your answer is no, then supply the appropriate justification. [2]

Question 6. Recall the following result discussed in class. Let F be the distribution function of a random vector taking values in $\mathbb{R}^2$. Then for real numbers $x_1' < x_1, x_2' < x_2$, we verified that

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - F\begin{pmatrix} x_1' \\ x_2 \end{pmatrix} - F\begin{pmatrix} x_1 \\ x_2' \end{pmatrix} + F\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \geq 0.$$

State and prove the corresponding version of this inequality in three dimensions. [5]