MID SEMESTER EXAMINATION (2019-20 ODD),
BASIC PROBABILITY AND DISTRIBUTION THEORY (MTH431A)
TOTAL MARKS: 32, MAXIMUM YOU CAN SCORE: 30

ROLL NO:

NAME:

Instructions:

(1) Illegible answers will be taken as incorrect.
(2) You may refer to your own class notes and use results proved in class. Searching in books/internet is not allowed. Mobiles must be switched off during the examination.
(3) At the end of the exam, return the question paper along with your answer-script.
(4) $\mathbb{N} = \{1, 2, \cdots\}$ denotes the set of natural numbers. $(\Omega, \mathcal{F}, \mathbb{P})$ denotes some arbitrary but fixed probability space.

1. Section A

Question 1. This question consists of MCQs with exactly one correct answer in each. Put a tick (✓) beside the correct option. Answer on top of this question paper. Make sure to return the question paper along with your answer-script at the end of the exam. You get no credit for rough work.

(I) Assume that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Consider the function $\mu : \mathbb{B}_{\mathbb{R}} \to [0, \infty]$ defined by

$$\mu(B) := \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \delta_n(B), \forall B \in \mathbb{B}_{\mathbb{R}}.$$

Then [2]

(a) μ is not a measure on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$.
(b) μ is not a probability measure on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$.
(c) There exists a real valued random variable with law μ.
(d) none of the above statements are true.

(II) Let $A, B \in \mathcal{F}$ be such that $\mathbb{P}(A) = \mathbb{P}(B) = 1$. Then $\mathbb{P}(A \cap B)$ equals [2]

$$(a)\ 0. \quad (b)\ \frac{1}{3}. \quad (c)\ \frac{1}{2}. \quad (d)\ 1.$$

(III) Let $2^{\mathbb{N}}$ denote the power set of $\mathbb{N}$, i.e. the set of all subsets of $\mathbb{N}$. Let μ be the counting measure defined on $(\mathbb{N}, 2^{\mathbb{N}})$. Consider the sets $A_n := \{n, n+1, n+2, \cdots\}, \forall n \in \mathbb{N}$. Then the values of $\lim_{n\to\infty} \mu(A_n)$ and $\mu\left(\bigcap_{n=1}^{\infty} A_n\right)$ are [3]

(a) 0 and 0 respectively. (b) ∞ and ∞ respectively. (c) ∞ and 0 respectively. (d) 0 and ∞ respectively.

Date: September 21, 2019. Time: 8:00 - 10:00 hrs. Venue: L1 ERES.

(IV) Let X and Y be two real valued random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Fix $A \in \mathcal{F}$. Consider the function $Z^A : \Omega \to \mathbb{R}$ defined by

$$Z^A(\omega) := \begin{cases} X(\omega), & \text{if } \omega \in A, \\ Y(\omega), & \text{otherwise.} \end{cases}$$

Then the statement 'Z^A is also a real valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$' is [3]

(a) true for all $A \in \mathcal{F}$. (b) false for all $A \in \mathcal{F}$. (c) may be true or false depending on $A \in \mathcal{F}$.

(V) Let X be a real valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose $\mathbb{E}X^2 = 4$. Then [2]

(a) $\mathbb{E}|X|$ need not be finite.
(b) $\mathbb{E}|X|$ exists and must take values in $[0, 2]$.
(c) $\mathbb{E}X$ exists and can take any value in $\mathbb{R}$.
(d) none of the above statements are true.

2. Section B

Instructions: Answer these questions in your answer-script. You can use any result proved in class.

Question 2. Let X be a real valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the following collection of subsets of Ω

$$\sigma(X) := \{X^{-1}(B) : B \in \mathbb{B}_{\mathbb{R}}\}.$$

(a) Show that $\sigma(X)$ is a σ-field of subsets of Ω. Provide all details. [4]
(b) Explicitly write down $\sigma(X)$, when X is a constant. Justify your answer. [1]

Question 3. While conducting a random experiment, a researcher found that the outcomes can be modelled by a real valued random variable X, such that jumps of its distribution function F occur exactly at the points a and b $(a < b)$. However, the continuous part of the distribution function could not be identified. Based on the experiments, the researcher also concluded that the events $(X = a)$ and $(X = b)$ are independent. Provide an example of such a random variable X or disprove the above conclusion. [5]

Question 4. (a) Define a median of a real valued random variable. [1]
(b) Consider all real valued random variables X with continuous probability density functions such that the mean, say μ, exists. Furthermore, assume that λ is a median of X. Is the inequality $\mathbb{E}|X - \lambda| \leq \mathbb{E}|X - \mu|$ true for all such X? Prove the statement or produce a counter-example. Justify your answer. [4]

Question 5. Fix $c > 0$. Consider a two-dimensional random vector $\binom{X}{Y}$ with joint probability density function $f : \mathbb{R}^2 \to [0, \infty)$ given by

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) := \frac{c}{2\pi} \frac{1}{(c^2 + x^2 + y^2)^{1.5}}, \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

(a) Compute the marginal probability density functions (PDFs), if they exist. [3]
(b) If the marginal PDFs exist, then verify that they are one-dimensional PDFs. If they do not exist, find the marginal distribution functions. [2]