ASSIGNMENT 1, MTH614A
DUE ON 14:00 HRS, FEBRUARY 14, 2019.

Instructions:

- Supply all details.
- You are encouraged to discuss with your classmates. However, write down the solutions on your own.
- In what follows, $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a probability space.
- Marks are indicated at the end of each problem. Total marks for this assignment is 5.

Problems:

Q1. (Jensen's inequality for Conditional Expectation) Let $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$, X a real valued, integrable random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\phi : \mathbb{R} \to \mathbb{R}$ a convex function such that $\mathbb{E}|\phi(X)| < \infty$. Show that a.s. [1]

$$\phi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\phi(X)|\mathcal{G}].$$

Q2. (Dominated Convergence Theorem for Conditional Expectation) Let $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$ and $\{X_n\}$ a sequence of real valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If there exists a non-negative and integrable random variable Y defined on the same probability space such that $|X_n| \leq Y, \forall n$, then show that [1]

$$\mathbb{E}[X_n|\mathcal{G}] \xrightarrow[a.s./\mathcal{L}^1(\mathbb{P})]{n\to\infty} \mathbb{E}[X|\mathcal{G}],$$

where X is the (a.s.) limit of the sequence $\{X_n\}$ (assume existence of the limit).

Q3. Suppose that two real valued stochastic processes $\{X_t\}$ and $\{Y_t\}$, defined on the same probability space, are modifications of each other. $[\frac{1}{2} + \frac{1}{2}]$

(a) If both $\{X_t\}$ and $\{Y_t\}$ have continuous paths, show that they are indistinguishable.

(b) If both $\{X_t\}$ and $\{Y_t\}$ have right (or left) continuous paths, show that they are indistinguishable.

Q4. Fix $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a locally α Hölder continuous function, for some $\alpha \in (0, 1]$. Show that f is β Hölder continuous for any $\beta \in (0, \alpha]$. $[\frac{1}{2}]$

Q5. Let $\{B_t\}$ be a Brownian motion started at $x \in \mathbb{R}$. Compute $\mathbb{E}(2(B_t - x))^n$ for all natural numbers n. [1]

Q6. Let $\{B_t\}$ be a standard Brownian motion and let $(\mathcal{F}_t)$ be the natural filtration of $\{B_t\}$. Show that $\{B_t^2 - t\}$ is an $(\mathcal{F}_t)$ martingale. $[\frac{1}{2}]$