ASSIGNMENT 2, MTH614A
DUE ON 14:00 HRS, MARCH 28, 2019.

Instructions:

- Supply all details.
- You are encouraged to discuss with your classmates. However, write down the solutions on your own.
- In what follows, $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a probability space.
- Marks are indicated at the end of each problem. Total marks for this assignment is 5.

Problems:

Q1. Let $\{f_n\}$ be a sequence of real valued continuous functions on $\mathbb{R}$ converging uniformly to a function $f : \mathbb{R} \to \mathbb{R}$. Show that f is continuous. $[\frac{1}{2}]$

Q2. From the definition of Itô integrals, verify the following two relations: $[\frac{1}{2} + \frac{1}{2}]$

(a) $\int_0^T B_s\, dB_s \overset{a.s.}{=} \frac{1}{2} B_T^2 - \frac{1}{2} T$.

(b) $\int_0^T s\, dB_s \overset{a.s.}{=} T B_T - \int_0^T B_s\, ds$.

Q3. Using the Itô formula, verify the following two relations: $[\frac{1}{2}]$

(a) $\int_0^T B_s\, dB_s \overset{a.s.}{=} \frac{1}{2} B_T^2 - \frac{1}{2} T$.

(b) $\int_0^T s\, dB_s \overset{a.s.}{=} T B_T - \int_0^T B_s\, ds$.

Q4. For any filtration $(\mathcal{F}_t)$, define

$$\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s, \forall t \geq 0.$$

Show that the filtration $(\mathcal{F}_{t+})$ is right continuous. $[\frac{1}{2}]$

Q5. For any sub σ-field $\mathcal{G}$ of $\mathcal{F}$, let $\overline{\mathcal{G}}$ denote the $\mathbb{P}$-completion of $\mathcal{G}$. Given any filtration $(\mathcal{F}_t)$, show that [1]

$$(\bar{\mathcal{F}}_{t+}) = (\overline{\mathcal{F}_{t+}}).$$

Q6. Given any $V \in \mathcal{L}(B)$, define a process $\{X_t\}$ by $X_t := \int_0^t V_s\, dB_s - \frac{1}{2} \int_0^t V_s^2\, ds$. Show that a.s. for all $t \geq 0$ $[\frac{1}{2}]$

$$\exp(X_t) = 1 + \int_0^t \exp(X_s) V_s\, dB_s.$$

Note: Processes of this type are called Exponential martingales.

Q7. In class, we have proved the Itô formula for any Itô process $\{X_t\}$ and any C_b^2 function $f : \mathbb{R} \to \mathbb{R}$. Using the same approach (Taylor's expansion), prove the Itô formula when $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is any $C_b^{1,2}$ function. [1]