ASSIGNMENT 3, MTH614A
DUE ON 9:00 HRS, APRIL 16, 2019.

Instructions:

- Supply all details.
- You are encouraged to discuss with your classmates. However, write down the solutions on your own.
- In what follows, $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ will denote a complete filtered probability space satisfying usual conditions.
- Marks are indicated at the end of each problem. Total marks for this assignment is 5.

Problems:

Q1. (Gronwall's inequality) Let $f : [0,T] \to \mathbb{R}$ be an integrable function such that $f(t) \leq \alpha + \beta \int_0^t f(s)\, ds, \forall t$, where α, β are non-negative constants. Prove that [$\frac{1}{2}$]

$$f(t) \leq \alpha \exp(\beta t), \forall t.$$

Remark: (not included in the exercise) if f is non-negative and $\alpha = 0$, then $f \equiv 0$.

Q2. Complete the following argument left out in class. Let $\sigma, b : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz continuous functions. Let $X_0 \in \mathcal{L}^2(\mathbb{P})$ and independent of a standard Brownian motion $\{B_t\}$. Using the solution $\{X_t^{(N)}\}$ of the SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)\, dt$$

constructed on the time interval $[0, N]$ (with the initial conditional X_0), construct a global solution, i.e. a solution on $[0, \infty)$. [1]

Q3. Continue with the same notations as in the previous exercise, but take σ and b to be locally Lipschitz continuous function. Prove the pathwise uniqueness of strong solutions as stated in class. [1]

Q4. Continue with the same notations and hypotheses as in $Q2$. Let $\{X_t\}$ be a strong solution. Find a upper bound of $\mathbb{E}X_t^2$ only involving σ, b and X_0. [1]

Q5. Prove that a process $\{X_t\}$ is an FV process if and only if it is locally FV. [$\frac{1}{2}$]

Q6. (a) Let $\{M_t\}$ be a local submartingale and $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function. Show that $\{\phi(M_t)\}$ is also a local submartingale. [$\frac{1}{2}$]

(b) Let $\{M_t\}$ be a square integrable martingale, i.e. $\mathbb{E}M_t^2 < \infty, \forall t$. Show that $t \mapsto \mathbb{E}M_t^2$ is a non-decreasing function on $[0, \infty)$. [$\frac{1}{2}$]