END SEMESTER EXAMINATION
INTRODUCTION TO STOCHASTIC CALCULUS MTH614A
VENUE: L13 ERES
DATE AND TIME: APRIL 26, 9:00 TO 12:00 HRS

General instructions:

- Each section below has specific instructions. Read them carefully.
- You can use your class notes. Searching in books/internet is not allowed.
- Maximum you can score: 40 (Answer as much as you can)
- Notation: In what follows, $\{B_t\}$ will denote a 1 dimensional standard Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1. Section A

Question 1. Write down the correct choice in the following questions. Each question has only one correct answer. You do not get any credit for the rough work.

(i) The limit $\limsup_{n\to\infty} n^{-\frac{1}{3}} B_n$ [2]

- is a.s. 0.
- is a.s. ∞.
- is a.s. $\frac{1}{3}$.
- does not exist.

(ii) Let $f : [0, 10] \to \mathbb{R}$ be a continuous function with finite variation. Then square variation of f on $[0, 10]$ [3]

- does not exist.
- is 0.
- is ∞.
- is the same as the total variation (as in finite variation).

(iii) Identify which one of the following processes is not a local martingale with respect to the natural filtration of $\{B_t\}$. [3]

- $\{B_{t\wedge 3}\}$.
- $\{\int_0^t \exp(B_s)\, dB_s\}$.
- $\{\exp(2(B_t - t))\}$.
- $\{B_t^2\}$

(iv) which of the following is NOT a standard Brownian motion? [3]

- $\{X_t\}$ where

$$X_t := \begin{cases} 0, \text{ if } t = 0, \\ tB_{\frac{1}{t}}, \text{ if } t > 0. \end{cases}$$

- $\{B_{t+2} - B_2\}_{t\geq 0}$.
- $\{\frac{1}{3} B_{3t}\}_{t\geq 0}$.
- $\{\frac{1}{3} B_{9t}\}_{t\geq 0}$.

(v) Let $\mathcal{C}$ be a sub σ-field of $\mathcal{F}$ and let $X, Y \in \mathcal{L}^1(\mathbb{P})$. Which one of the following is NOT a property of the Conditional Expectation? [3]

- $\mathbb{E}[3X + Y \mid \mathcal{C}] \overset{a.s.}{=} 3\mathbb{E}[X \mid \mathcal{C}] + \mathbb{E}[Y \mid \mathcal{C}]$.
- $\mathbb{E}[\mathbb{E}[Y \mid \mathcal{C}]] = \mathbb{E}Y$.
- If $\sigma(X)$ and $\mathcal{C}$ are independent, then $\mathbb{E}[X \mid \mathcal{C}] \overset{a.s.}{=} \mathbb{E}X$.
- If X is measurable with respect to $\mathcal{C}$, then $\mathbb{E}[X \mid \mathcal{C}] \overset{a.s.}{=} \mathbb{E}X$.

(vi) Let $\{X_t\}$ be the unique strong solution to the equation

$$dX_t = dB_t - 2(X_t + B_t)\, dt; X_0 = 1.$$

Suppose Q is an equivalent probability measure such that $\{B_t^*\}$ is a Brownian motion under Q and

$$dX_t = dB_t^* - 2X_t\,dt; X_0 = 1.$$

Apply Girsanov Theorem to identify which one of the following could be the density $\frac{dQ}{d\mathbb{P}}$ on the given filtration $(\mathcal{F}_t)$? [3]

- $\exp(2\int_0^t B_s\,dB_s - 2\int_0^t B_s^2\,ds)$
- $\exp(-2\int_0^t B_s\,dB_s - 2\int_0^t B_s^2\,ds)$
- $\exp(2\int_0^t B_s\,dB_s - t^2\,ds)$
- $\exp(2\int_0^t B_s\,dB_s - 2t)$

(vii) Let $\{X_t\}$ be a strong solution of the SDE $dX_t = \sigma(X_t)dB_t + b(X_t)\,dt; X_0 = 1$, where $\sigma, b : \mathbb{R} \to \mathbb{R}$ are continuous functions. Which one of the following is NOT true? [3]

- If σ and b are globally Lipschitz, then strong solutions are pathwise unique.
- If σ and b are locally Lipschitz, then strong solutions are pathwise unique.
- If σ and b are globally Lipschitz, then X_1 has moments of all orders.
- $\{X_t\}$ is adapted to the filtration of $\{B_t\}$.

2. Section B

Instruction: You may use any result proved in class.

Question 2. Is $\sup_{s\in[0,2]} B_s$ absolutely continuous? Justify your answer – compute the density function or otherwise, disprove the fact. [4]

Question 3. Fix $T > 0$. From the definition of Itô integrals, verify the following relation: [6]

$$\int_0^T s\,dB_s \overset{a.s.}{=} TB_T - \int_0^T B_s\,ds$$

Question 4. Fix $T > 0$. Consider the following sequence of Stochastic processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$:

$$X_t^{(1)} = 3, \forall t \in [0, T]$$

and for $n = 1, 2, \cdots$

$$X_t^{(n+1)} = 3 + \int_0^t (X_s^{(n)} - 1)\,ds + 2\int_0^t \sin(X_s^{(n)})\,dB_s, \forall t \in [0, T].$$

Check whether the sequence converges in $\mathcal{L}^2([0, T] \times \Omega, Leb \otimes \mathbb{P})$. Here Leb denotes the Lebesgue measure restricted to the interval $[0, T]$. [6]

Question 5. Solve the following SDE and find the unique strong solution $\{X_t\}$:

$$dX_t = 2\,dB_t - 3X_t\,dt; X_0 = 1.$$

Also find $\mathbb{E}X_t$ and $Var(X_t)$. [3 + 1 + 1]

Question 6. Consider the evolution of stock price $\{S_t\}$ given by $S_t = S_0 \exp(Y_t), t \geq 0$ where

$$dY_t = \mu\,dt + \sigma\,dB_t; Y_0 = 0$$

and μ, σ are the drift coefficient and volatility of the market respectively. Let r denote the risk-free interest rate and $F(S_t, t)$ denote the price of a call option at time t. Deduce the Black-Scholes PDE. [10]