MID SEMESTER EXAMINATION
INTRODUCTION TO STOCHASTIC CALCULUS MTH614A
VENUE: L11 ERES
DATE AND TIME: FEBRUARY 23, 15:30 TO 17:30 HRS

General instructions:

- Each section below has specific instructions. Read them carefully.
- No notes or books are allowed during the exam.
- Maximum you can score: 30
- Notation: Unless stated otherwise, $\{B_t\}$ will denote a standard Brownian motion (1 dimensional) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathbb{B}_{\mathbb{R}^n}$ denotes the Borel σ-field on $\mathbb{R}^n$.

1. Section A

Question 1. Write down ALL correct choices in the following questions. You do not get any credit for the rough work. [$1\times 5 = 5$]

(i) Fix $t \geq 0$. Then a.s.
  (a) both $\limsup_{n\to\infty} \frac{B_{t+n}-B_t}{\sqrt{n}}$ and $\liminf_{n\to\infty} \frac{B_{t+n}-B_t}{\sqrt{n}}$ are finite.
  (b) $\limsup_{n\to\infty} \frac{B_{t+n}-B_t}{\sqrt{n}}$ is finite, but $\liminf_{n\to\infty} \frac{B_{t+n}-B_t}{\sqrt{n}} = -\infty$
  (c) $\limsup_{n\to\infty} \frac{B_{t+n}-B_t}{\sqrt{n}} = \infty$, but $\liminf_{n\to\infty} \frac{B_{t+n}-B_t}{\sqrt{n}}$ is finite.
  (d) $\limsup_{n\to\infty} \frac{B_{t+n}-B_t}{\sqrt{n}} = \infty$ and $\liminf_{n\to\infty} \frac{B_{t+n}-B_t}{\sqrt{n}} = -\infty$.

(ii) Quadratic variation of $\{B_t\}$
  (a) does not exist.
  (b) is 0 a.s..
  (c) is t a.s..
  (d) is $\sqrt{t}$ a.s..

(iii) For any $t > 0, s > 0$, the conditional expectation $\mathbb{E}[(B_{2t+s} - B_{t+2s})^2 | B_s]$
  (a) does not exist.
  (b) $\overset{a.s.}{=} t - s$.
  (c) $\overset{a.s.}{=} |t - s|$.
  (d) $\overset{a.s.}{=} (t - s)^2$.

(iv) Let $\{X_t\}$ be a 2 dimensional standard Brownian motion. Let $\{X_t^{(1)}\}$ and $\{X_t^{(2)}\}$ denote the one dimensional co-ordinate processes. Then the law of the random variable $X_1^{(1)} + 3X_2^{(2)}$
  (a) can not be determined from the given information.
  (b) is $N(0, 7)$.
  (c) is $N(0, 19)$.
  (d) is χ_7^2.

(v) Let $\{Y_t\}$ and $\{Z_t\}$ be two continuous modifications of a real valued stochastic process $\{X_t\}$. Then
  (a) $\mathbb{P}(Y_t = Z_t) = 1, \forall t$.
  (b) $\mathbb{P}(Y_t = Z_t, \forall t) = 1$.
  (c) $\{\alpha Y_t + (1-\alpha) Z_t\}$ is also a continuous modification of $\{X_t\}$ for any $\alpha \in \mathbb{R}$.

2. Section B

Instructions: You need to answer only using the first principles (such as definitions). Marks will be deducted unless all the steps have been successfully explained. However, you are allowed to use the following results/facts without any further explanation, 1) existence of a Brownian Motion 2) Strong Markov property of Brownian Motion and 3) basic results from Measure Theoretic Probability (as explained in class).

Question 2. Let X be a real valued, integrable random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{D}$ denote a sub σ-field on $\mathcal{F}$ and $c > 0$ be a constant. Prove the following variant of conditional Markov inequality: a.s. [4]

$$\mathbb{P}(|X| \geq c \mid \mathcal{D}) \leq \frac{1}{c^3}\mathbb{E}[|X|^3 \mid \mathcal{D}].$$

Question 3. Let $\{X_t\}_{t\in[0,\infty)}$ be a real valued continuous, adapted process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Further, assume that $\{X_t\}$ is a centred Gaussian process with $Cov(X_s, X_t) = s \wedge t, \forall s, t \in [0, \infty)$. Show that $\{X_t\}$ is a standard Brownian motion. [4]

Question 4. Let $(\mathcal{F}_t)$ denote the natural filtration of $\{B_t\}$ and $c > 0$ be a constant. Show that the random time [4]

$$\tau(\omega) := \inf\{t > 0 : |B_t(\omega)| \geq c\}$$

is an $(\mathcal{F}_t)$ stopping time.

3. Section C

Instruction: You may use any result proved in class.

Question 5. (a) Let $\{X_t\}$ be a real valued continuous, adapted process. Show that the map $I : \Omega \to \mathbb{R}$ defined by $\omega \mapsto \int_1^2 X_s(\omega)\, ds$ is $\mathcal{F}_2/\mathbb{B}_{\mathbb{R}}$ measurable. [3]

(b) Find the variance of $\int_1^2 B_s\, ds$. [2]

(c) Using the reflection principle proved in class, write down the distribution of $\inf_{t\in[0,T]} B_t$ for any fixed $T > 0$. [2]

Question 6. (a) Consider the time inversion $\{X_t\}$ of $\{B_t\}$, i.e.

$$X_t := \begin{cases} 0, & \text{if } t = 0, \\ tB_{\frac{1}{t}}, & \text{if } t > 0. \end{cases}$$

Show that $\{X_t\}$ is a Gaussian process. [4]

(b) Using the first part or otherwise, prove the following statement: [3]

$$\lim_{t\to\infty} \frac{B_t}{t} \overset{a.s.}{=} 0.$$

Question 7. (a) Compute the 4-th central moment of a standard Normal random variable. [3]

(b) State the Kolmogorov-Centsov Theorem. [2]

(c) Using parts (a) and (b), what can you say about the paths of $\{B_t\}$? [2]

4. Section D

Question 8. Consider the random time $\tau : \Omega \to [0, \infty]$ defined by

$$\tau(\omega) := \sup\{t \in [0, 1] \mid B_t(\omega) = 0\}.$$

Using the reflection principle of Brownian motion, prove the Lévy's arc-sine Law: [10]

$$\mathbb{P}(\tau \leq t) = \frac{2}{\pi}\sin^{-1}(\sqrt{t}), t \in [0, 1].$$

Hint: $\mathbb{P}(\tau \leq t) = \mathbb{P}(B_s \neq 0, \forall s \in [t, 1]) = \int_{-\infty}^{\infty} \mathbb{P}(B_s \neq 0, \forall s \in [t, 1] \mid B_t = x)\, \mathbb{P} \circ B_t^{-1}(dx)$