ASSIGNMENT 1, MTH754A
DUE ON 9:00 HRS, AUGUST 16, 2018.

Instructions:

- Supply all details.
- You are encouraged to discuss with your classmates. However, write down the solutions on your own.
- In what follows, Ω will always denote some non-empty set.
- Marks are indicated at the end of each problem. Total marks for this assignment is [5].

Problems:

Q1. Suppose that a collection $\mathcal{F}$ of subsets of Ω has the following properties.
 (a) $\emptyset \in \mathcal{F}$.
 (b) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
 (c) If $A \in \mathcal{F}$, then A^c is a finite disjoint union of members of $\mathcal{F}$.
Consider another collection $\mathcal{C}$ consisting of finite disjoint union of members of $\mathcal{F}$. Show that $\mathcal{C}$ is a field of subsets of Ω. $\left[\frac{1}{2}\right]$
Remarks: (not included in the exercise) The collection of left-open right-closed intervals in $\mathbb{R}$ is an example of such an $\mathcal{F}$. Recall that we consider intervals of the form $(a, \infty), a \in \mathbb{R}$ to be left-open right-closed. The collection $\mathcal{C}$ was mentioned in the class as an example of a field, but not a σ-field.

Q2. Show that open sets in $\mathbb{R}$ can be written as a disjoint union of finitely (or countably) many open intervals. [1]
Remarks: (not included in the exercise) This fact was used in the class to show the equality of the Borel σ-field $\mathbb{B}_{\mathbb{R}}$ generated by the open sets in $\mathbb{R}$ and the σ-field generated by the open intervals.

Q3. Let $f : \Omega \to \Omega'$ be a function and let $\mathcal{F}$ be a σ-field in Ω'. Show that $f^{-1}(\mathcal{F}) = \{f^{-1}(A) : A \in \mathcal{F}\}$ is a σ-field in Ω. $\left[\frac{1}{2}\right]$

Q4. Let $f : \Omega \to \Omega'$ be a function and let $\mathcal{F}$ be a collection of subsets of Ω'. Show that

$$\sigma(f^{-1}(\mathcal{F})) = f^{-1}(\sigma(\mathcal{F})),$$

where $f^{-1}(A)$ denotes the pre-image of $A \subset \Omega'$, $f^{-1}(\mathcal{F}) = \{f^{-1}(A) : A \in \mathcal{F}\}$ and $f^{-1}(\sigma(\mathcal{F})) = \{f^{-1}(A) : A \in \sigma(\mathcal{F})\}$. $\left[\frac{1}{2}\right]$ (Hint: Use Q3 and the good sets principle.)

Q5. Say that a σ-field is *countably generated (or separable)* if it can be generated by some countable collection of sets. Show that the Borel σ-field $\mathcal{B}_{\mathbb{R}}$ on $\mathbb{R}$ is countably generated. $\left[\frac{1}{2}\right]$

Q6. Suppose that $\mathcal{F}$ is a collection of subsets of Ω with the following properties: $\Omega \in \mathcal{F}$ and $A, B \in \mathcal{F}$ implies $A \setminus B := A \cap B^c \in \mathcal{F}$. Show that $\mathcal{F}$ is a field. $\left[\frac{1}{2}\right]$

Q7. Suppose that a collection $\mathcal{F}$ of subsets of Ω is both a π system and a λ system. Show that $\mathcal{F}$ is a σ-field. $\left[\frac{1}{2}\right]$

Q8. (Uniqueness of extension) Let $\mathcal{F}$ be a π-system on Ω. Suppose that $\mathbb{P}_1$ and $\mathbb{P}_2$ are two probability measures defined on the measurable space $(\Omega, \sigma(\mathcal{F}))$ such that they agree on $\mathcal{F}$, i.e.

$$\mathbb{P}_1(A) = \mathbb{P}_2(A), \forall A \in \mathcal{F}.$$

Show that the measures agree on $\sigma(\mathcal{F})$. [1]