# ASSIGNMENT 2, MTH754A
# DUE ON 12:00 HRS, AUGUST 28, 2018.

**Instructions:**

- Supply all details.
- You are encouraged to discuss with your classmates. However, write down the solutions on your own.
- Marks are indicated at the end of each problem. Total marks for this assignment is [5].
- $(\Omega, \mathcal{F})$ will denote a measurable space.
- $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$

**Problems:**

Q1. Construct a $\sigma$-finite measure space $(\Omega, \mathcal{F}, \mu)$ with the following property: there exists a sequence $\{A_n\}$ such that $A_n \downarrow \emptyset$, but $\lim_n \mu(A_n) = \infty \neq \mu(\emptyset)$. $\left[\frac{1}{2}\right]$

Q2. Let $(\Omega_i, \mathcal{F}_i), i = 1, 2$ be measurable spaces and let $\mathcal{A}$ be a collection of subsets of $\Omega_2$ which generates $\mathcal{F}_2$. Assume that $f : \Omega_1 \to \Omega_2$ satisfies $f^{-1}(A) \in \mathcal{F}_1, \forall A \in \mathcal{A}$. Show that $f$ is measurable. $\left[\frac{1}{2}\right]$

Q3. Show that all continuous maps $f : \mathbb{R}^n \to \mathbb{R}^m$ are Borel measurable. $\left[\frac{1}{2}\right]$ (Hint: Use Q2)

Q4. Let $\{f_n\}$ be a sequence of $\bar{\mathbb{R}}$ valued Borel measurable functions defined on $\Omega$.

(a) If $\lim_n f_n(\omega)$ exists for all $\omega \in \Omega$, then show that the function $\omega \mapsto \lim_n f_n(\omega)$ is an $\bar{\mathbb{R}}$ valued Borel measurable function. $\left[\frac{1}{2}\right]$

(b) Show that the functions $\omega \mapsto \limsup_n f_n(\omega)$ and $\omega \mapsto \sup_n f_n(\omega)$ are $\bar{\mathbb{R}}$ valued Borel measurable functions. $\left[\frac{1}{2} + \frac{1}{2}\right]$

Remark: (not included in the exercise) Similar arguments apply for $\liminf_n f_n$ and $\inf f_n$. Note that $\limsup_n f_n, \liminf_n f, \sup_n f_n, \inf_n f_n$ always exist, but $\lim_n f_n$ may not.

Q5. Let $f$ and $g$ be $\bar{\mathbb{R}}$ valued Borel measurable functions on $(\Omega, \mathcal{F})$. Fix $A \in \mathcal{F}$ and define the function $h : \Omega \to \bar{\mathbb{R}}$ by

$$h(\omega) = \begin{cases} f(\omega), & \omega \in A, \\ g(\omega), & \omega \in A^c. \end{cases}$$

Show that $h$ is also Borel measurable. $\left[\frac{1}{2}\right]$

Q6. Let $X, Y : \Omega \to \mathbb{R}$ be Borel measurable. Assume that $Y$ is $\sigma(X)/\mathbb{B}_{\mathbb{R}}$ measurable. Show that there exists a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f \circ X$. [1]

Remark: (not included in the exercise) If $X : \Omega \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are measurable, then so is $f \circ X$. This is the converse of Q6.

Q7. Let $\{A_n\}$ be a sequence of subsets of $\Omega$. For all $\omega \in \Omega$, prove the following statements.

$$\liminf_{n\to\infty} 1_{A_n}(\omega) = 1_{\liminf_{n\to\infty} A_n}(\omega),$$

and

$$\limsup_{n\to\infty} 1_{A_n}(\omega) = 1_{\limsup_{n\to\infty} A_n}(\omega). \quad \left[\frac{1}{2}\right]$$