ASSIGNMENT 3, MTH754A
DUE ON 9:00 HRS, SEPTEMBER 10, 2018.

Instructions:

- Supply all details.
- You are encouraged to discuss with your classmates. However, write down the solutions on your own.
- In what follows, $(\Omega, \mathcal{F}, \mu)$ will denote a measure space.
- Marks are indicated at the end of each problem. Total marks for this assignment is [5].

Problems:

Q1. Let $f : \Omega \to \mathbb{R}$ be a non-negative, Borel measurable function. Fix $A \in \mathcal{F}$. Show that

$$\int_A f\, d\mu = \sup\left\{\int_A s\, d\mu : 0 \leq s \leq f, s \text{ simple}\right\}. \left[\frac{1}{2}\right]$$

Q2. Let $f : \Omega \to \mathbb{R}$ be a non-negative, Borel measurable, integrable function. Consider the set function $\nu : \mathcal{F} \to [0, \infty]$ defined by

$$\nu(A) := \int_A f(\omega)\, \mu(d\omega) = \int_\Omega f(\omega) 1_A(\omega)\, \mu(d\omega),\ \forall A \in \mathcal{F}.$$

Show that $(\Omega, \mathcal{F}, \nu)$ is a finite measure space. [1]

Q3. Let f, g be Borel measurable, integrable functions such that

$$\int_A f\, d\mu \leq \int_A g\, d\mu,\ \forall A \in \mathcal{F}.$$

Show that $f \leq g$, μ-a.e.. $\left[\frac{1}{2}\right]$

Q4. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable and fix $a \in \mathbb{R}$. Consider the two integrals '$\int_\mathbb{R} f(x)\, dx$' and '$\int_\mathbb{R} f(x-a)\, dx$' with respect to the Lebesgue measure. If one integral exists, show the existence of the other. In this case, show that the two integrals are actually equal. $\left[\frac{1}{2} + \frac{1}{2}\right]$

Q5. Prove the following version of Markov inequality. Let $f : \Omega \to \bar{\mathbb{R}}$ be Borel measurable. Then for any $A \in \mathcal{F}$ and $c > 0$ show that

$$\mu(\{|f| \geq c\} \cap A) \leq \frac{1}{c}\int_A |f|\, d\mu. \left[\frac{1}{2}\right]$$

Q6. Construct a probability space $(\Omega, \mathcal{F}, \mu)$ and a real valued integrable function f on this space such that f is not bounded. $\left[\frac{1}{2}\right]$

Q7. Fix $a, b \in \mathbb{R}$ with $a < b$. Suppose that there exist functions $f : (a, b) \times \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ such that

(a) $\omega \in \Omega \mapsto f(t, \omega)$ is μ-integrable for every fixed $t \in (a, b)$,
(b) $t \in (a, b) \mapsto f(t, \omega)$ is continuous for every fixed $\omega \in \Omega$,
(c) $g \geq 0$ and μ-integrable,
(d) $|f(t, \omega)| \leq g(\omega), \forall t, \omega$.

Then show that the function $h : (a, b) \to \mathbb{R}$ defined by

$$h(t) := \int_\Omega f(t, \omega)\, d\mu(\omega)$$

is continuous. $\left[\frac{1}{2}\right]$

Q8. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) := \frac{1}{\pi}\frac{1}{1+t^2}$. Show that

$$\int_\mathbb{R} t^+ f(t)\, dt = \int_\mathbb{R} t^- f(t)\, dt = \int_\mathbb{R} |t| f(t)\, dt = \infty. \left[\frac{1}{2}\right]$$

Remark: Soon we shall encounter the concept of density of a random variable. The above result will then be restated as follows: the mean of a random variable with density f (a Cauchy random variable) does not exist.