ASSIGNMENT 4, MTH754A
DUE ON 9:00 HRS, OCTOBER 11, 2018.

Instructions:

- Supply all details.
- You are encouraged to discuss with your classmates. However, write down the solutions on your own.
- In what follows, $(\Omega, \mathcal{F}, \mu)$ will denote a measure space.
- Marks are indicated at the end of each problem. Total marks for this assignment is [5].

Problems:

Q1. In class, while proving a result on the characterization of Riemann integrability, we have left out an argument involving discontinuity points of a given function. This argument may be done in two similar lines of attack. Complete the argument.

(a) Let $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable. Let $\{\mathcal{P}_k\}$ be a sequence of partitions such that $\mathcal{P}_k \subseteq \mathcal{P}_{k+1}$ and $|\mathcal{P}_k| \to 0$. Recall the functions $\overline{f^{\mathcal{P}_k}}, \underline{f^{\mathcal{P}_k}}, \overline{f}, \underline{f}$ which were defined in class. If x is not the end-point of any subintervals of the partitions $\mathcal{P}_k$, then show that

$$f \text{ is continuous at } x \in (a, b) \iff f(x) = \overline{f}(x) = \underline{f}(x). \ [\tfrac{1}{2}]$$

(b) Given f as above, define two functions $g, h : [a, b] \to \mathbb{R}$ by

$$g(x) := \lim_{\delta \downarrow 0} \inf_{y:|y-x|\leq\delta} f(y), \ \ h(x) := \lim_{\delta \downarrow 0} \sup_{y:|y-x|\leq\delta} f(y).$$

Show that

$$f \text{ is continuous at } x \in (a, b) \iff f(x) = g(x) = h(x). \ [\tfrac{1}{2}]$$

Q2. In class, we have seen the correspondence between the set of Lebesgue-Stieltjes measures on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ and the set of distributions functions on $\mathbb{R}$. Later, we started proving the same correspondence in $\mathbb{R}^2$. Complete the argument (you may follow the arguments as in dimension one).

(a) (from measures to distributions functions) Let μ be a Lebesgue-Stieltjes measure on $(\mathbb{R}^2, \mathbb{B}_{\mathbb{R}^2})$. Define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F((x_1, x_2)) := \mu(\{(a, b) \in \mathbb{R}^2 : -\infty < a \leq x_1, -\infty < b \leq x_2\}).$$

Show that F is a distribution function on $\mathbb{R}^2$, i.e. it is 'right-continuous' and 'non-decreasing' (as per the definition given in class). $[\frac{1}{2}]$

(b) (from distributions functions to measures) Let F be a distribution function on $\mathbb{R}^2$. Construct a Lebesgue-Stieltjes measure on $(\mathbb{R}^2, \mathbb{B}_{\mathbb{R}^2})$, by extending the set function μ such that $\mu((x_1, y_1] \times (x_2, y_2]) := F(y_1, y_2) - F(x_1, y_2) - F(y_1, x_2) + F(x_1, x_2)$. [1]

Remark: (not included in the exercise) Compare the previous relation with the inclusion-exclusion principle.

Q3. Let μ be a Lebesgue-Stieltjes measure on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ such that it is translation invariant on intervals, i.e. for any real number $x \in \mathbb{R}$ and any interval I in $\mathbb{R}$, we have $\mu(I) = \mu(x + I)$. Show that there exists a non-negative constant c such that $\mu(A) = c\lambda(A), \forall A \in \mathbb{B}_{\mathbb{R}}$, where λ denotes the Lebesgue measure on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$. $[\frac{1}{2}]$

Q4. (a) Let $f, g : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ be measurable functions. Fix real numbers $p, q \in (1, \infty)$ such that $p^{-1} + q^{-1} = 1$. Then, prove the following version of Hölder's inequality, viz.

$$\int_\Omega |fg| \, d\mu \leq \left(\int_\Omega |f|^p \, d\mu\right)^{1/p} \left(\int_\Omega |g|^q \, d\mu\right)^{1/q}. \ [\tfrac{1}{2}]$$

(b) Let X be a real valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As an application of the first part, show that

$$\mathbb{E}|X| \leq \left(\mathbb{E}|X|^2\right)^{1/2}. \ [\tfrac{1}{2}]$$

Q5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For every sequence $\{A_n\}$ of sets in $\mathcal{F}$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \implies \mathbb{P}(\limsup_n A_n) = 0. \ [\tfrac{1}{2}]$$

Remark: This result is usually referred to as Borel-Cantelli Lemma first half (or direct half). This is useful in practice and allows us to construct/identify explicit null sets.

Q6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a measurable space. Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are two probability measures on this space. Show that that the following conditions are equivalent. $[\tfrac{1}{2}]$

(a) For any $A \in \mathcal{F}$, $\mathbb{P}_1(A) = 0$ implies $\mathbb{P}_2(A) = 0$.

(b) For every $\epsilon > 0$, there exists $\delta > 0$ such that if $A \in \mathcal{F}$ with $\mathbb{P}_1(A) < \delta$, then $\mathbb{P}_2(A) < \epsilon$.

Remark: We shall see the importance of the above equivalence later, when we study a property of a pair of measures, viz. the absolute continuity.