ASSIGNMENT 5, MTH754A
DUE ON 9:00 HRS, NOVEMBER 1, 2018.

Instructions:

- Supply all details.
- You are encouraged to discuss with your classmates. However, write down the solutions on your own.
- In what follows, $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a probability space.
- Marks are indicated at the end of each problem. Total marks for this assignment is [5]. $\left[\frac{1}{2}\right]$

Problems:

Q1. Find examples of random variables $X, X_1, X_2, \cdots$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the indicated properties hold. $\left[4 \times \frac{1}{2}\right]$

(a) $X_n \xrightarrow[n\to\infty]{\mathcal{L}^1(\mathbb{P})} X$, but $X_n \nrightarrow X$ a.s.

(b) $X_n \xrightarrow[n\to\infty]{\text{a.s.}} X$, but $X_n \nrightarrow X$ in $\mathcal{L}^1(\mathbb{P})$.

(c) $X_n \xrightarrow[n\to\infty]{\text{in probability}} X$, but $X_n \nrightarrow X$ in $\mathcal{L}^1(\mathbb{P})$.

(d) $X_n \xrightarrow[n\to\infty]{\text{in probability}} X$, but $X_n \nrightarrow X$ a.s.

Q2. Prove the following convergence: $N(0, \frac{1}{n}) \Rightarrow \delta_0$, where $N(\mu, \sigma^2)$ denotes the law of a Gaussian random variable on $(\mathbb{R}, \mathbb{B}_\mathbb{R})$ with mean μ and variance σ^2. $\left[\frac{1}{2}\right]$

Q3. Let $\{x_n\}$ be a sequence of real numbers. Find a necessary and sufficient condition such that $\delta_{x_n} \Rightarrow \delta_0$. $\left[\frac{1}{2}\right]$

Q4. Let $X, X_1, X_2, \cdots$ be random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove the following equivalences. $\left[\frac{3}{2}\right]$

$$
\begin{aligned}
& X_n \xrightarrow[n\to\infty]{\text{in probability}} X. \\
\iff\ & \mathbb{E}\frac{|X_n - X|}{1 + |X_n - X|} \xrightarrow{n\to\infty} 0. \\
\iff\ & \mathbb{E}\left(|X_n - X| \wedge 1\right) \xrightarrow{n\to\infty} 0. \\
\iff\ & \inf\{\epsilon > 0 : \mathbb{P}(|X_n - X| > \epsilon) < \epsilon\} \xrightarrow{n\to\infty} 0.
\end{aligned}
$$

Note: One can now formulate metrics on the set of random variables, the convergence in which are equivalent to the convergence in probability. These metrics are sometimes referred to as the Ky Fan metrics.

Q5. Let $X_1, X_2, \cdots$ be random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose $X_n \Rightarrow c$, where $c \in \mathbb{R}$ is some constant. Show that $X_n \xrightarrow[n\to\infty]{\text{in probability}} c$. $\left[\frac{1}{2}\right]$