ASSIGNMENT 6, MTH754A
DUE ON 9:00 HRS, NOVEMBER 15, 2018.

Instructions:

- Supply all details.
- You are encouraged to discuss with your classmates. However, write down the solutions on your own.
- In what follows, $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a probability space.
- Marks are indicated at the end of each problem. Total marks for this assignment is [5].

Problems:

Q1. Let $\{X_n\}$ be a sequence of real valued random variables on a probability space $(\Omega, \mathcal{F})$, such that the sequence converges in probability to a random variable X. Show that there exists a subsequence $\{X_{n_k}\}$ converging a.s. to X. $\left[\frac{1}{2}\right]$.

Q2. Let $\{F_n\}$ be a sequence of distribution functions (of probability measures) on $\mathbb{R}$. Show that there exists a subsequence $\{F_{n_k}\}$ and a non-decreasing, right continuous function F such that $\lim_k F_{n_k}(x) = F(x)$ at all continuity points of F. Prove or disprove: F is a distribution function of a probability measure on $\mathbb{R}$. $\left[\frac{1}{2}\right]$

Remark: This result is usually referred to as Helly's selection theorem.

Q3. Let ν and μ be measures on $(\Omega, \mathcal{F})$ with $\nu \ll \mu$. Show that $\int g d\nu = \int g f d\mu$ for all real valued measurable functions g, where $f = \frac{d\nu}{d\mu}$. $\left[\frac{1}{2}\right]$.

Q4. Let $X \in \mathcal{L}^2(\mathbb{P})$ and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Then for any $\mathcal{G}$ measurable random variable Y, show that

$$\mathbb{E}(X - Y)^2 \geq \mathbb{E}\left(X - \mathbb{E}[X|\mathcal{G}]\right)^2. \left[\frac{1}{2}\right]$$

Q5. Fix $p \in [0, 1]$. Consider a sequence of independent random variables $\{X_n\}$ such that

$$X_n = \begin{cases} \frac{1}{n}, & \text{w.p. } p \\ -\frac{1}{n}, & \text{w.p. } 1 - p \end{cases}$$

Find all values of p such that the series $\sum_n X_n$ converges a.s.. [1]

Q6. Let X and Y be independent random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Further assume that $X, Y \in \mathcal{L}^2(\mathbb{P})$. Show that $X, Y, X^2, Y^2, XY \in \mathcal{L}^1(\mathbb{P})$ and that

$$\mathbb{E}(XY) = (\mathbb{E}X)(\mathbb{E}Y), Var(X + Y) = Var(X) + Var(Y).[1]$$

Q7. In each of the following cases, find the pointwise (i.e. almost sure) limit of $\frac{S_n}{n}$ as $n \uparrow \infty$, where $S_n := |X_1| + \cdots + |X_n|$ and $\{X_n\}$ is an iid sequence of random variables with

(a) law $\frac{1}{2}N(0, 1) + \frac{1}{2}\delta_0$. $\left[\frac{1}{2}\right]$

(b) law given by the density function $\frac{1}{\pi(1+x^2)}, x \in \mathbb{R}$ (Cauchy distribution). $\left[\frac{1}{2}\right]$