END SEMESTRAL EXAMINATION
PROBABILITY THEORY (MTH754A)
DATE AND TIME: NOVEMBER 22, 16:00 TO 19:00 HRS

General instructions:

- Each section below has specific instructions. Read them carefully.
- You can carry only the assignments (without solutions) in the exam hall. However, no notes or books are allowed during the exam.
- Start answering a problem on a new page.
- Maximum you can score: 25
- Notation: $\mathbb{B}_{\mathbb{R}^d}$ denotes the Borel σ-field on $\mathbb{R}^d$, generated by the open sets. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ denotes an arbitrary probability space.

1. Section A

Question 1. Write down ALL correct choices in the following questions. You do not get any credit for rough work. You receive 1 for a correct answer, 0 for a partially correct answer and -1 for a wrong answer. [1×5 = 5]

(i) Consider measures μ on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ such that $\mu(\{0\}) = 0$ and $\int_{\mathbb{R}} (|x|^2 \wedge 1)\ \mu(dx) < \infty$. Such measures are known as Lévy measures. The statement 'Lévy measures are σ-finite' is [1]
   (a) true.
   (b) false.

(ii) Let f be a real valued, bounded, measurable function defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose there exists a function $g : \Omega \to \mathbb{R}$ such that $f = g$ a.s.. Then [1]
   (a) g need not be measurable.
   (b) g is measurable, but need not be integrable.
   (c) g is measurable and integrable.
   (d) $g \in \bigcap_{p \in [1,\infty]} \mathcal{L}^p(\mathbb{P})$.

(iii) Suppose that X and Y are independent random variables on a common probability space with $X \sim N(0,1), Y \sim N(1,1)$. Compute $\mathbb{E}(3Y - 5X^3Y^2)$. [1]
   (a) 0
   (b) -1
   (c) 3
   (d) does not exist

(iv) Let *Leb* and $N(0,1)$ denote the Lebesgue measure and the standard Gaussian measure on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ respectively. Then [1]
   (a) Both $N(0,1) \ll Leb$ and $Leb \ll N(0,1)$ are false.
   (b) $N(0,1) \ll Leb$ is true, but $Leb \ll N(0,1)$ is false.
   (c) $N(0,1) \ll Leb$ is false, but $Leb \ll N(0,1)$ is true.
   (d) Both $N(0,1) \ll Leb$ and $Leb \ll N(0,1)$ are true.

(v) Let Φ be a characteristic function of a real valued random variable. Consider the complex conjugate $\bar{\Phi}$ of Φ. The statement '$\bar{\Phi}$ is also a characteristic function' is [1]
   (a) true.
   (b) false.

2. Section B

Instruction: You need to answer only using the first principles (such as definitions). Marks will be deducted unless all the steps have been successfully explained. However, you are allowed to use the following facts without any further explanation, 1) generating sets of $\mathbb{B}_{\mathbb{R}}$ and 2) $\mu(\emptyset) = 0$, if μ is a measure.

Question 2. Let $\{A_n\}$ be a sequence of events in $(\Omega, \mathcal{F}, \mathbb{P})$. Prove Boole's inequality, i.e. [2]

$$\mathbb{P}\left(\bigcup_n A_n\right) \leq \sum_n \mathbb{P}(A_n).$$

Question 3. Let μ and ν be two probability measures on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ with the same distribution function. Show that $\mu = \nu$. [2]

Question 4. Let $\{X_n\}$ be a sequence of i.i.d random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = \frac{1}{2}, \forall n \geq 1$. Let $S_n := X_1 + \cdots + X_n$ and $\mathcal{F}_n := \sigma(X_1, \cdots, X_n), \forall n \geq 1$. Compute $\mathbb{E}[S_{n+1}|\mathcal{F}_n] - S_n$. (You may assume the Radon-Nikodym Theorem) [2]

Question 5. Let $\{f_n\}$ be a sequence of real valued measurable functions on a measure space $(\Omega, \mathcal{F}, \mu)$ such that

$$\sup_n \int_\Omega |f_n|^{1+\epsilon}\, d\mu < \infty$$

for some $\epsilon > 0$. Show that the sequence is uniformly integrable, i.e. the following holds [2]

$$\lim_{x\to\infty} \sup_n \int_{(|f_n|>x)} |f_n|\, d\mu = 0.$$

3. Section C

Instruction: You may use any result proved in class, but not the assignments.

Question 6. Answer one of the following. If both are answered, then only part (a) will be evaluated.

(a) Let $\mathcal{F}$ be a field of subsets of a non-empty set Ω. Show that the minimal σ field $\sigma(\mathcal{F})$ containing $\mathcal{F}$ and the minimal monotone class $\mathcal{M}(\mathcal{F})$ containing $\mathcal{F}$ coincide. [4]

(b) Let $\{X_n\}$ be a sequence of non-negative, integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

$$\mathbb{E}\left(\sum_n X_n\right) = \sum_n \mathbb{E}X_n.$$

Does the result hold if $\mathbb{E}X_n = \infty$ for some n? [3 + 1]

Question 7. Fix $1 \leq p < q < \infty$. Let X be a real valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

$$(\mathbb{E}|X|^p)^{\frac{1}{p}} \leq (\mathbb{E}|X|^q)^{\frac{1}{q}}.$$

Let $\{X_n\}$ be a sequence of random variables on the same probability space. Suppose $\{X_n\}$ converges to X in $\mathcal{L}^p(\mathbb{P})$. Does it converge in $\mathcal{L}^q(\mathbb{P})$? What can you say about the converse case? [2 + 1 + 1]

Question 8. Let $\{X_n\}$ be a sequence of real valued, i.i.d random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Fix $x \in \mathbb{R}$. Define another sequence $\{Y_n\}$ of random variables by

$$Y_n := 1_{(X_n \leq x)}, \forall n.$$

Find the distribution of Y_n. Also find the mean and variance of Y_n. Are they i.i.d? Find the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n Y_n$, if it exists. [1 + 1 + 1 + 1]

4. Section D

Instruction: You may use any result proved in class or given in the assignments. You may answer at most one question.

Question 9. Let Leb_d denote the Lebesgue measure on $(\mathbb{R}^d, \mathbb{B}_{\mathbb{R}^d})$. Consider $\lambda_d := Leb_d(B_d(0,1))$, where $B_d(0,1)$ denotes the open unit ball in $\mathbb{R}^d$ (with respect to the usual Euclidean metric) centred at the origin. Check if $\lim_{d\to\infty} \lambda_d$ exists and if so, find the value. [4]

Question 10. Let $\{X_n\}$ be a sequence of independent random variables (real valued) on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $S_n := X_1 + \cdots + X_n, n \geq 1$. If the sequence $\{S_n\}$ converges in probability, show that it also converges almost surely. [4]

Question 11. Let X and Y be i.i.d random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with mean 0 and variance 1. If $X + Y$ and $X - Y$ are independent, show that the distribution of X (and Y) are $N(0, 1)$. [4]