MID SEMESTRAL EXAMINATION
PROBABILITY THEORY MTH754A
VENUE: L12 ERES
DATE AND TIME: SEPTEMBER 22, 15:30 TO 17:30 HRS

General instructions:

- Each section below has specific instructions. Read them carefully.
- You can carry only the assignments (without solutions) in the exam hall. This is for your reference if you wish to answer the question from Section D (see below).
- No notes or books are allowed during the exam.
- Start answering a problem on a new page.
- Maximum you can score: 20
- Notation: $\mathbb{B}_{\mathbb{R}^d}$ denotes the Borel σ-field on $\mathbb{R}^d$, generated by the open sets.

1. Section A

Question 1. Write down ALL correct choices in the following questions. You do not get any credit for the rough work. [$1\times4 = 4$]

(i) Suppose that $\mathcal{C}$ is a collection of subsets of a non-empty set Ω and $\mathcal{E}$ is a σ-field containing $\mathcal{C}$. Then $\sigma(\mathcal{C}) \subseteq \mathcal{E}$ holds provided [1]
  (a) $\mathcal{C}$ is a field.
  (b) $\mathcal{C}$ is a π-system.
  (c) $\mathcal{C}$ is a Monotone class.

(ii) Let $f : (\mathbb{R}, \{\emptyset, \mathbb{R}\}) \to (\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ be a measurable function. Then [1]
  (a) f is a non-constant differentiable function.
  (b) f is continuous.
  (c) f is a constant function.

(iii) Consider the Cosine function $\cos : \mathbb{R} \to \mathbb{R}$. Then the statement 'The function cos can be uniformly approximated by simple functions on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$' is [1]
  (a) true.
  (b) false.

(iv) Let μ_1 and μ_2 be a finite measure and a σ-finite measure on the measurable space $(\Omega, \mathcal{F})$ respectively. Given a non-negative measurable function $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ consider the set function $\lambda : \mathcal{F} \to [0, \infty]$ defined by

$$\lambda(A) := \int_A f \, d\mu_1 + \int_A f \, d\mu_2.$$

Then [1]
  (a) λ is a finite measure.
  (b) λ is a σ-finite measure.
  (c) none of the above.

2. Section B

Instruction: You need to answer only using the first principles (such as definitions). Marks will be deducted unless all the steps have been successfully explained. However, you are allowed to use the following facts without any further explanation, 1) generating sets of $\mathbb{B}_{\mathbb{R}}$ and 2) $\mu(\emptyset) = 0$, if μ is a measure.

Question 2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. Fix $A \in \mathcal{F}$. Show that

$$\mathcal{C} := \{B \in \mathcal{F} : \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)\}$$

is a Monotone class. [4]

Question 3. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ be a measure space and let $f : (\Omega_1, \mathcal{F}_1) \to (\Omega_2, \mathcal{F}_2)$ be a measurable function. Show that the push-forward set function $\mu_2 := \mu_1 \circ f^{-1}$ defined by

$$\mu_1 \circ f^{-1}(A) := \mu_1(\{\omega_1 \in \Omega_1 : f(\omega_1) \in A\}), \ \forall A \in \mathcal{F}_2.$$

is a measure on $(\Omega_2, \mathcal{F}_2)$. [4]

Question 4. Let μ be a measure on the measurable space $(\Omega, \mathcal{F}, \mu)$. Let $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ be a non-negative integrable function. Show that $\int_A f \, d\mu = 0$ whenever $\mu(A) = 0, A \in \mathcal{F}$. [4]

3. Section C

Instruction: You may use any result proved in class, but not the assignments.

Question 5. Prove the following variant of Markov inequality. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f : \Omega \to \mathbb{R}$ be Borel measurable. Then for every $c > 0$ show that [4]

$$\mu(\{|f| \geq c\}) \leq \frac{1}{c^2} \int |f|^2 \, d\mu.$$

Question 6. Compute the value of [4]

$$\lim_{n\to\infty} \int_0^1 \frac{1 + nx^2}{(1 + x^2)^n} \, dx.$$

Question 7. Let λ denote the Lebesgue measure on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$. For any Borel set A and $x \in \mathbb{R}$, define $A + x := \{a + x : a \in A\}$. Show that $A + x$ is again a Borel set and that λ is translation invariant, i.e. [1 + 3 = 4]

$$\lambda(A) = \lambda(A + x), \ \forall x \in \mathbb{R}, A \in \mathbb{B}_{\mathbb{R}}.$$

4. Section D

Instruction: You may use any result proved in class or given in the assignments.

Question 8. Let μ be a measure on $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ with the property that $\mu(A) < \infty$ for any bounded Borel set A. Then for any Borel set B, show that there exist a closed set C and an open set O such that $C \subseteq B \subseteq O$ and $\mu(O \setminus C) = 0$. [4]