ChE677 Introduction to Polymer Physics and Rheology

Assignment 1 **Due Date: 23 August 2006**

1. Show that $\langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - (\langle x \rangle)^2$ for any random variable x.

2. Consider the Poisson distribution

$$P(m) = \frac{a^m \exp[-a]}{m!}$$

where $m = 0, 1, 2, \cdots$.

   (a) Show that this distribution is normalized.
   (b) Calculate the mean $\langle m \rangle$.
   (c) Calculate the variance $\langle (m - \langle m \rangle)^2 \rangle$, and show that for a Poisson distribution the variance is equal to the mean.

3. Consider the Gaussian distribution:

$$P(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp[-\frac{(x - \langle x \rangle)^2}{2\sigma^2}], -\infty \le x \le \infty$$

By carrying out the required integrations explicitly:

   (a) Show that $\int_{-\infty}^{\infty} dx\, P(x) = 1$.
   (b) Calculate the 'n-th central moment' defined by $\langle (x - \langle x \rangle)^n \rangle$ for $n = 1, 2, 3$ and 4.

4. Let x be a random variable such that $0 \le x \le \infty$, and let $P(x)$ be its probability distribution function. The characteristic function $\phi(s)$ of the probability distribution is defined as

$$\phi(s) = \int_0^\infty dx \, \exp[-sx] P(x)$$

Find a relation between $\phi(s)$ and the moments of $P(x)$.

Hint: Expand the exponential inside the integral and proceed.

5. Consider the Binomial distribution discussed in the lecture.

$$P(x) = \frac{n!}{x!(n-x)!} p^x q^{n-x} \tag{1}$$

where p and q are the probabilities of success and failures at each trial, and $P(x)$ is the probability of obtaining x successes in n trials.

This problem guides you to derive the Gaussian distribution as a limiting case of the Binomial distribution in the limit $n \gg 1$, $x \gg 1$ and finite p and q.

Note the Stirling's approximation for $n \gg 1$: $n! \approx \exp[-n] n^n \sqrt{2\pi n}$.

Approximate the factorials by Stirling's approximation and simplify to get:

$$P(x) \approx \left(\frac{np}{x}\right)^x \left(\frac{nq}{n-x}\right)^{n-x} \sqrt{\frac{n}{2\pi x(n-x)}} \tag{2}$$

Show that if $\delta = x - np$, then $x = np + \delta$ and $n - x = nq - \delta$. Make these substitutions in the above equation. To evaluate the first two factors in $P(x)$ (ignore the square root for now): Take logarithm of the first two factors; show that

$$\log \frac{np}{x} = -\log\left(1 + \frac{\delta}{np}\right)$$

and a similar formula for $\log[nq/(n-x)]$; expand the logarithms in a series of powers in $\delta/(np)$, collect terms and simplify to get:

$$\log\left[\left(\frac{np}{x}\right)^x\left(\frac{nq}{n-x}\right)^{n-x}\right]\approx-\frac{\delta^2}{2npq}$$

Now exponentiate this again to get an approximation for the first two factors in Eq. 2. Now assemble an expression for $P(x)$, and verify that $P(x)$ indeed converges to the Gaussian in the large-n limit.

6. Consider the Binomial distribution given in Eq. (1). This problem will guide you to obtain the Poisson distribution in the limit $n\to\infty$, $p\to 0$ and $np\to$ finite.

   First, show that $n!/(n-x)!\approx n^x$ for fixed x and large n. *Hint:* Write $n!/(n-x)!$ as a product of x factors, divide by n^x and show that the limit is 1 as $n\to\infty$.

   Then write $q^{n-x}=(1-p)^{n-x}=\left(1-\frac{np}{n}\right)^n(1-p)^{-x}$; evaluate the limit of the first factor as $n\to\infty$, np fixed. The limit of the second factor as $p\to 0$ is 1. Assemble your results, and verify that you obtain a Gaussian. *Hint:* In the limit $n\to\infty$, np finite, show that, by expanding in a series:

$$\left(1-\frac{np}{n}\right)^n\approx\exp[-np]$$