Convergence Rate Analysis of Consensus Algorithms for Large-Scale Wireless Sensor Networks

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CERTIFICATE

It is certified that the work contained in the thesis entitled **"Convergence Rate Analysis of Consensus Algorithms for Large-Scale Wireless Sensor Networks"** being submitted by **Mr. V Sateeshkrishna Dhuli** has been carried out under my supervision. In my opinion, the thesis has reached the standard fulfilling the requirement of regulation of the Ph.D. degree. The results embodied in this thesis have not been submitted elsewhere for the award of any degree or diploma.

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Synopsis

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Wireless sensor network (WSN) consists of a large number of tiny sensor nodes with the limited power, computation, and wireless communication capabilities. Sensor nodes will be deployed to monitor an environment or detect an event in unattended and hostile environments. A typical task of the WSN is to measure some parameter of interest such as humidity, temperature, pressure, etc. In centralized algorithms, sensor nodes measure such parameters independently and delivery data to the fusion center. The fusion center gathers data from different sensor nodes and makes the final decisions. However, these algorithms require the proper organization of the sensor nodes and sophisticated routing protocols to forward the data to the fusion center. Fusion centers should be equipped with high energy and computational resources which increases the cost of the large-scale WSNs. Node or link failures in WSNs lead to frequent topology changes, and centralized algorithms are less resilient to topology changes. Because of these reasons, these algorithms are practically inefficient for WSN applications.

Consensus algorithms have attracted a lot of attention in the last two decades, due to their ability to compute the desired global statistics by exchanging the information only within the direct neighbors. In contrast to centralized algorithms, the underlying distributed and decentralized philosophy avoids the need for a fusion center for gathering the data. Primarily, these algorithms are suitable in the following situations: 1) global network topology information is not known; 2) dynamic topology changes because of the node or link failures, and network consists of resource constrained nodes. Hence, consensus algorithms are more appealing to WSN scenarios.

However, consensus algorithms are inherently iterative, and the graph Laplacian eigenvalues characterize the convergence rate. Consequently, estimating the convergence rate of consensus algorithms is a computationally challenging task for large-scale WSNs. Theoretical results play a significant role in the initial stage of WSN design and also reliable than simulation-based studies. Although there have been several works on consensus algorithms, the effect of WSN parameters on the convergence rate of consensus algorithms is yet to be investigated. To gain the advantages of the consensus algorithms for WSNs, it is essential to study the convergence of consensus algorithms for large-scale WSNs and examine the effect of network parameters on the convergence rate.

This thesis studies the convergence rate of the consensus algorithms for r-nearest neighbor networks and one-dimensional lattice networks. These network models represent the notion of geographical proximity in the wireless sensor networks and facilitate the closed-form expressions of convergence rate for the consensus algorithms. One-dimensional lattice networks and r-nearest neighbor networks allow the theoretical analysis that incorporates essential parameters like connectivity, scalability, network size, transmission radius, link failures, asymmetric links and network dimension.

The first part of the thesis considers the undirected graph modeling and models the WSN as an r-nearest neighbor ring, r-nearest neighbor torus, m-dimensional r-nearest neighbor torus networks and derives the generalized expressions of convergence rate for average consensus algorithms. This part studies the problem of the estimating convergence rate of the average consensus algorithms for large-scale WSNs. Further, the analytical expressions of convergence rate are derived in terms of the number of nodes, transmission radius, and network dimension.

The second part of this thesis considers the directed graph modeling and models the WSN as a ring, r-nearest neighbor ring, torus, and m-dimensional torus networks. Further, we derive the closed-form expressions of the convergence rate for average consensus algorithms. Subsequently, the effect of asymmetric links on the convergence rate of average consensus algorithms has been studied. Further, this part examines the absolute error introduced by the assumption of undirected graph modeling in analyzing the average consensus algorithms.

Gossip algorithm is an asynchronous version of consensus algorithm, where the global statistics will be computed using the local pair-wise communications. WSN has been modeled as a one-dimensional lattice network and derived the closed-form expressions of convergence rate for average periodic gossip algorithms. Next, using linear weight updated approach, a generalized analytic expression for convergence rate has been derived in terms of gossip weight and the number of nodes. Further, considering the link failures in WSNs, we obtain the explicit expressions of convergence rate for average periodic gossip algorithms in terms of the number of nodes and probability of link failures. Finally, the effect of the node's transmission radius on the convergence rate of periodic gossip algorithms has been studied using power-iteration and deflation techniques.

Dedicated to

my grand mother

Ginjala Ravanamma

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(V Sateeshkrishna Dhuli)

List of Symbols

G	Graph.
V	Set of Vertices.
E	Set of Edges.
d	Degree of the node.
λ	Eigenvalue.
m	Network dimension.
p	Probability of link failures.
L	Laplacian Matrix.
A	Adjacency Matrix.
D	Degree Matrix.
W	Weight Matrix.
a_{ij}	Entries of the Adjacency Matrix.
L_{ij}	Entries of the Laplacian Matrix.
h	Consensus Parameter.
γ	Convergence Parameter.
R	Convergence Rate.
r	Nearest neighbors.
w	Gossip weight.
n	Number of nodes.
i	Complex number.
a	Asymmetric link factor.
P	Power Consumption.
x	State variable.
k_1	Number of nodes in a torus network.
k_2	Number of nodes in a torus network.
E_1	Periodic subsequence.
υ	Eigen vector.
C_{k_1}	Ring Network.
T_{k_1,k_2}	Torus Network.
I	Identity Matrix.
M	Stochastic Matrix.
p_{ij}	Entries of Stochastic Matrix.
α	Path-loss Exponent.

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Chapter 1

Introduction

Wireless Sensor Networks(WSNs) have been utilized for the numerous applications such as battlefield surveillance, target tracking, environmental monitoring, health care, and disaster management (see, e.g., [1], [2], [3], [4], [5]). A typical task of the WSN is to measure some parameter of interest such as humidity, temperature, pressure, etc. In centralized algorithms, sensor nodes sense and measure such parameters independently and delivery data to the fusion center. Fusion centers gather the data from different sensor nodes and make the final decisions. However, these algorithms require the proper organization of the sensor nodes and sophisticated routing protocols to forward the data to the fusion center. Fusion centers should be equipped with high energy and computational resources which increases the cost of the large-scale WSNs. Node or link failures in WSNs lead to frequent topology changes, and centralized algorithms are less resilient to topology changes. Hence, these algorithms are not suitable for WSN applications. Consensus algorithms have attracted a lot of attention in the last two decades [6], [7], [8], [9], [10], [11], [12] due to their ability to compute the desired global statistics by exchanging the information only with the direct neighbors. In contrast to centralized algorithms, the underlying distributed and decentralized philosophy avoids the need for a fusion center for gathering the data. This thesis modeled the WSN as an r-nearest neighbor and one-dimensional lattice networks and derived the closed-form expressions of convergence rate for consensus algorithms. In this chapter, we briefly discuss the wireless sensor networks and present their applications in various areas. We also discuss the advantages and disadvantages of centralized and distributed consensus algorithms for wireless sensor network scenarios. Further, the problem statement of this thesis work has been presented.

1.1 Wireless Sensor Networks

A WSN consists of a large number of low-cost tiny sensor nodes to measure the properties of the environment such as temperature, pressure, humidity, light, etc. The number of sensor nodes in WSNs may be on the order of thousands to millions depending on the applications. A typical WSN is as shown in the Fig. 1.1. Sensor nodes sense, measure, and gather information from the environment and transmit the sensed data to the base station. Fig. 1.2 shows the schematic diagram of the major components of a sensor node. A sensor node can collect the data and forward it to either specific destination or the other sensors in the vicinity. In [1], [2], authors described the WSN protocol stack and discussed the open research problems in WSNs. In [3], authors surveyed the operating system, network services, applications, and deployment issues in WSNs. Sensor nodes are small in size with the limited computational, sensing, and wireless communication capabilities. These nodes may fail due to lack of power or environmental issues. Therefore, routing protocols in WSNs should be designed to improve the WSN's performance with limited resources. In [13], [14], authors presented a detailed survey of routing protocols for WSNs.

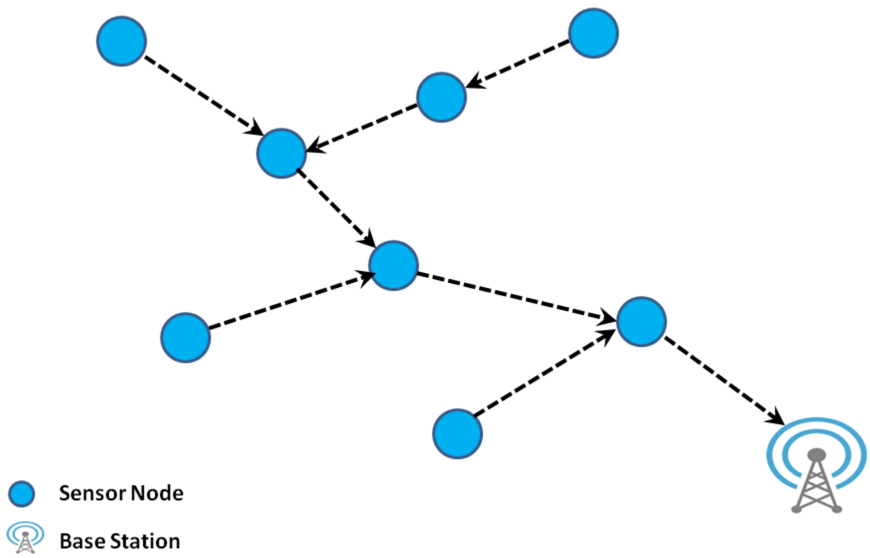

Figure 1.1: Typical Wireless Sensor Network

Applications

WSNs consist of different type of sensors such as seismic, thermal, visual, infrared, acoustic to measure the various parameters that include the temperature, humidity, vehicular movement, lighting condition, pressure, soil quality, noise levels, etc. [1]. The salient features of the WSNs lead to the wide range of applications that includes health, military, disaster management, managing inventory, and monitoring product quality [4]. Environmental applications of sensor networks include tracking the movements of birds, animals, insects, monitoring the crops, irrigation, and forest fire detection, etc. Health applications of sensor networks provides patient monitoring, drug inspection, and surveillance of hospital premises (e.g., [1], [2],[3], [5]). In [4], [5] authors presented a detailed survey on the applications of WSNs in various areas.

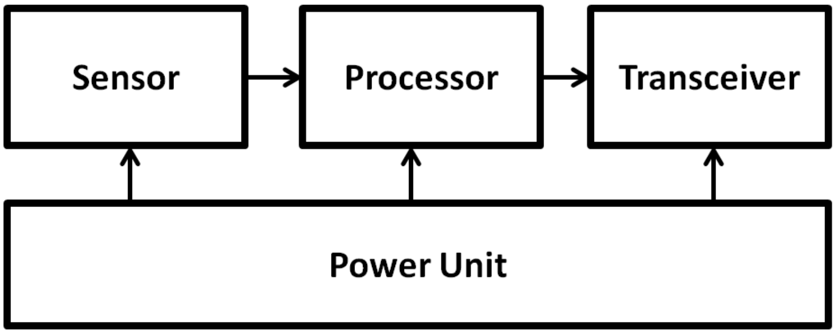

Figure 1.2: Major Components of a Sensor Node

Power Consumption

Sensor nodes are equipped with the limited power resources and hence WSN's lifetime strongly depends on sensor nodes' battery lifetime [1], [3]. Therefore, power conservation is a significant concern in the design of the routing algorithms for WSNs [13], [14]. The primary task of a sensor node in a WSN is to detect the events, perform the data processing, and then transmit the data to either other sensor nodes or base station. WSNs are prone to node or link failures due to lack of power resources or physical damage. Node or link failures can cause significant network topological changes and may affect the WSN performance. Sensor nodes consume power during sensing, communication, and data processing operations. Nodes consume a substantial amount of power in transmitting the data over sensing, processing, and receiving the data.

Centralized Algorithms

In centralized algorithms, sensor nodes form the clusters and transmit the data to a cluster head as shown in the Fig. 1.3. Fusion center (FC) Receives data from cluster heads and make final decisions on the received data [15]. Here, FCs are less robust to node or link failures and responsible for the reorganization of the WSNs. As FCs need to aggregate the data, they need to be equipped with the more power and computational resources over the normal sensor nodes. These algorithms suffer from the following disadvantages [16], [17]: (1) Power consumption and communication overhead is significant increases in large-scale networks; (2) Need of sophisticated routing protocols which are robust against the topology changes; (3) Fusion centers are expensive among the normal nodes, hence cost of the network drastically increases in large-scale WSNs; (4) Failure of cluster heads/Fusion centers lead to the failure of WSN services. Thus, centralized algorithms are highly inefficient in terms of cost, energy consumption, and scalability. Authors discussed the energy efficient centralized algorithms in [16]. In [18], authors presented the advantages and disadvantages of centralized algorithms. In [19], authors surveyed the recent work on data aggregation algorithms and discussed the main features, advantages, and disadvantages.

1.2 Distributed Consensus Algorithms

In distributed consensus algorithms [20], nodes can organize themselves and perform the computations with the neighboring nodes as shown in the Fig. 1.4. Every node communicates the data with the direct neighbors and makes the decisions without the need of any fusion center. Here, every sensor node needs to perform the sensing, communicating, and aggregating operations. These algorithms are suitable in the following situations: 1) global network topology information is not known; 2)dynamic topology changes because of frequent node failures; 3)network consists of resource constrained nodes. Hence, these algorithms are more appealing to wireless sensor network (WSN)

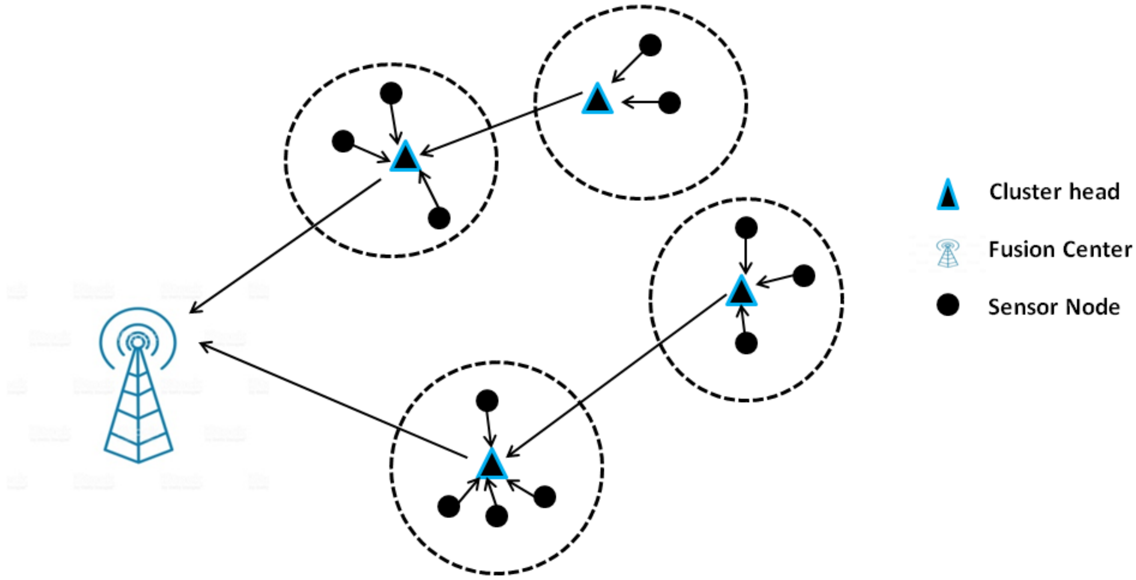

Figure 1.3: Centralized Algorithms in WSN

scenarios [21], [22], [23].

1.3 Problem Statement

Consensus algorithms are inherently iterative, and the convergence rate characterizes their performance. The second largest eigenvalue of the weight matrix determines the convergence rate of average periodic gossip algorithms. Computing eigenvalues for large-dimensional matrices require sophisticated algorithms and high-performance computing resources. Hence, it is difficult to predict the convergence rate of these algorithms for large-scale networks. In this thesis, we study the convergence rate of the consensus

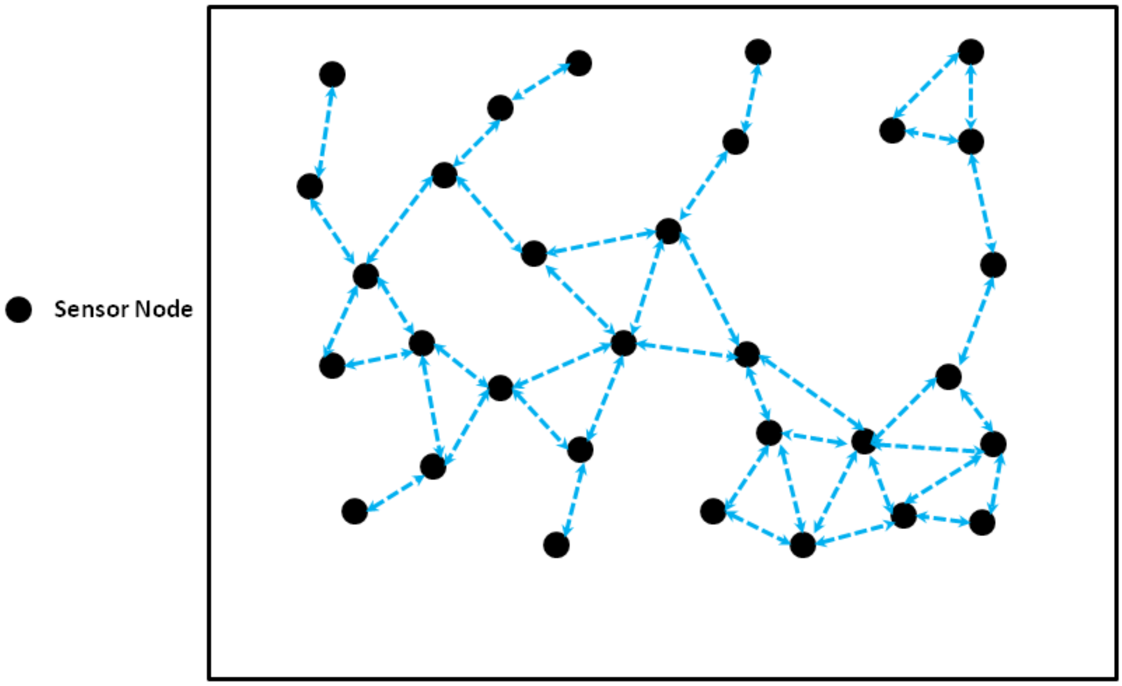

Figure 1.4: Distributed Algorithms in WSN

algorithms for large-scale WSNs in terms of network parameters. We model the WSN as an r-nearest neighbor network and one-dimensional lattice network and derive the closed-form expressions of the convergence rate for average consensus algorithms and periodic gossip algorithms. Theoretical results developed in this thesis reduce the computational complexity drastically and also provides important insights for the design of consensus algorithms on WSNs.

1.4 Thesis Contributions and Organization

The thesis has been organized in the following six chapters.

Chapter 2 reviews the distributed consensus algorithms. The theory of the average consensus algorithms has been presented. After that, gossip algorithms and periodic gossip algorithms have been explored. Finally, this chapter explains the convergence rate of the consensus algorithms.

In **chapter 3**, WSN has been modeled as an r-nearest neighbor ring, r-nearest neighbor torus, and m-dimensional r-nearest neighbor torus network and derived the explicit expressions of convergence rate for average consensus algorithms. In this chapter, WSN has been modeled as an undirected graph, and the effect of the number of nodes, transmission radius and network dimension on convergence rate has been studied.

Chapter 4 studies the effect of asymmetric links on the convergence rate. In this chapter, WSN has been modeled as a directed graph, and the analytic expressions of convergence rate of average consensus algorithms for the ring, torus, r-nearest neighbor ring, and m-dimensional torus networks have been derived.

In **chapter 5**, WSN has been modeled as a one-dimensional lattice network, and the closed-form expressions of convergence rate for average periodic gossip algorithms have been derived. Next, using the linear weight update approach, a generalized analytic expression for convergence rate has been derived in terms of gossip weight and the number of nodes. Further, explicit expressions of convergence rate of average periodic gossip algorithms have been obtained in terms of the number of nodes and probability of link failures. Finally, numerical results have been presented to investigate the effect of the number of nodes, gossip weight, and the probability of link failures on the convergence rate of periodic gossip algorithms.

In **chapter 6**, WSN has been modeled as an r-nearest neighbor ring network, and the effect of node's transmission radius on the convergence rate of periodic gossip algorithms has been studied. This chapter proposes the power-iteration and deflation techniques to evaluate the convergence rate of periodic gossip algorithms.

Finally, **chapter 7** presents the conclusions of the thesis and discusses the problems which can be further studied.

Chapter 2

Distributed Consensus Algorithms

This chapter reviews the graph theory concepts and presents the important definitions in graph theory. Further, we discuss the average consensus algorithms and defines the convergence rate. Finally, this chapter reviews the gossip algorithms and also explains the periodic gossip algorithms.

2.1 Graph Theory Concepts

In this thesis, the information flow in the WSNs has been modeled by a graph. The vertices of a graph represent the sensor nodes, and edges represent the connectivity among them. Laplacian matrix describes the connectivity of the network. Graph Laplacian eigenvalues play a vital role in characterizing the convergence rate or convergence time of consensus algorithms. Spectral graph theory studies the graph Laplacian eigenvalues. This chapter presents the fundamental concepts of graph theory [24], [25] and introduces the notations used in the forthcoming chapters.

Definitions

Basic notations and definitions of graph theory concepts will be presented in this section.

Definition 2.1.1. *A graph $G = (V, E)$ is a set of vertices V with the set of edges E connecting some of the vertices.*

Definition 2.1.2. *The degree of a vertex υ is defined as the number of edges which connect to the vertex.*

Definition 2.1.3. *Let $G = (V, E)$ be a graph and $\upsilon \in V$ represents a node. If edge $e = \{\upsilon\}$, then edge is called self loop.*

Definition 2.1.4. *Two vertices υ_1 and υ_2 are said to be adjacent if there exists an edge $e \in E$ such that $e = \{\upsilon_1, \upsilon_2\}$.*

Definition 2.1.5. *Two edges e_1 and e_2 are said to be adjacent if there exists a vertex υ so that υ is an element of both the edges.*

Definition 2.1.1. *Network Diameter of a graph is defined as the shortest distance of two most distant vertices. Diameter of the graph can be computed as the longest of all the shortest path lengths obtained from every vertex to all other vertices. It measures the number of steps needed for convergence in distributed consensus algorithms [8], [11].*

Example 2.1.1. *Let us consider the graph in 2.1. The set of vertices for this graph can be written as $V = \{1, 2, 3, 4, 5\}$. The set of edges can be written as*
$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}$. Here, total number of edges and nodes are 5 and degree of each node is 2.

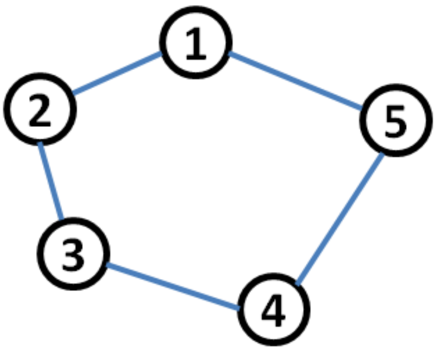

Figure 2.1: Graph

Definition 2.1.6. *The edge set consists of unordered pairs in a undirected graph and ordered pairs in a directed graph. As shown in the 2.2, undirected graph and directed graph model the symmetric and asymmetric network links respectively.*

Definition 2.1.7. *In a directed graph, in-degree of a vertex* υ *in* G *is the total number of edges in* E *with the destination* υ*. Out-degree of* υ *is the total number of edges in* E *with the source* υ*.*

Definition 2.1.8. *The adjacency matrix (*A*) of a graph* G *is the matrix that consists of the following entries as*

$$a_{ij} = \begin{cases} 1 & if \quad (i,j) \in \varepsilon \\ 0 & Otherwise \end{cases} \tag{2.1.1}$$

Definition 2.1.9. *The degree matrix (*D*) of a graph* G *is the matrix that consists of the following entries as*

$$d_{ij} = \begin{cases} d_i & if \quad i = j \\ 0 & Otherwise \end{cases}$$

where d_i *is the degree of a vertex* i*.*

Definition 2.1.10. *The Laplacian matrix (*L*) describes the connectivity of the network.*

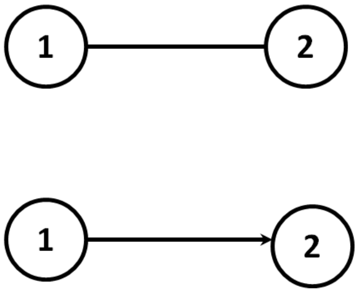

Figure 2.2: Example of undirected and directed graphs

This matrix consists of following entries

$$L_{ij} = \begin{cases} d_i & if \quad i = j \\ -1 & if \quad i \neq j \\ 0 & otherwise \end{cases}$$

where d_i *is the degree of a vertex 'i'. The Laplacian matrix can be written as* $L = D - A$.

Example 2.1.2. *For the Fig. 2.1, adjacency matrix can be written as*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Degree matrix of the Fig. 2.1 is written as

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Finally, we can write the Laplacian matrix of the Fig. 2.1 as

$$L = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Properties of Laplacian Matrix

Laplacian matrix plays a major role in the distributed algorithms. Especially, convergence rate of the distributed algorithms is characterized by the graph Laplacian eigenvalues. The properties of the Laplacian matrix summarized below [26]:

(1)L is a symmetric positive semidefinite matrix.

(2)The off-diagonal entries of 'L' are -1.

(3)The diagonal entries of 'L' are the vertex degrees and the row sums and the column sums are all zero.

(4)The eigenvalues of a Laplacian matrix are as follows

$$0 = \lambda_1(L) \leq \lambda_2(L) \leq \lambda_3(L) \ldots\ldots\ldots \leq \lambda_N(L).$$

(5)The graph topology is connected only if its zero eigenvalue has multiplicity one. The second smallest eigenvalue $\lambda_2(L) > 0$ is the algebraic connectivity or the fiedler value [27] of the network.

Standard Graph Models

In this section, some of the basic graph models will be discussed with examples [25]. These models will be extremely useful in studying the real time networks such as wireless sensor networks, adhoc networks, delay-tolerant networks. etc.

Regular Graphs

Let $G = (V, E)$ be a graph with $|V| = n$. If the degree sequence of G is $(l, l, l, ...l)$ with $l \leq (n-1)$, then 'G' is called a l-regular graph.

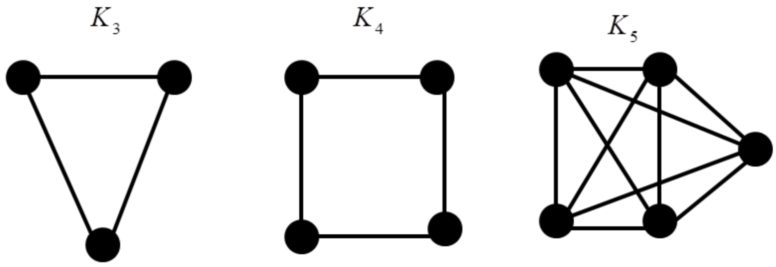

Figure 2.3: Complete Graph

Complete Graphs

The *complete graph* is the undirected graph of *'n'* vertices whose edge set consists every possible edge as shown in the 2.3. The edge and vertex sets in a complete graph are as follows

$$V = \{\nu_1, \nu_2, ...\nu_n\}$$

$$E = \{(\nu_j, \nu_k) \,|1 \leq j \leq (n-1), (j+1) \leq k \leq n\}$$
$$|E| = \frac{n(n-1)}{2}$$

Path Graphs

Path graphs are formed by stringing *'n'* vertices together in a path structure as shown in the 2.4. It has vertex and edge sets as follows

$$V = \{\nu_1, \nu_2, ...\nu_n\}$$

$$E = \{(\nu_j, \nu_{j+1}) \,|1 \leq j \leq n-1\}$$

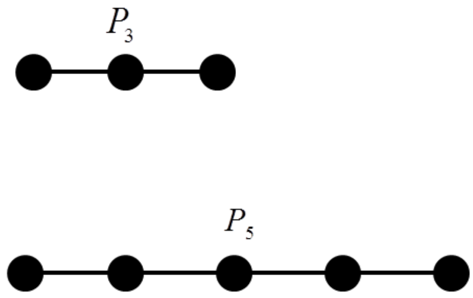

Figure 2.4: Path Graph

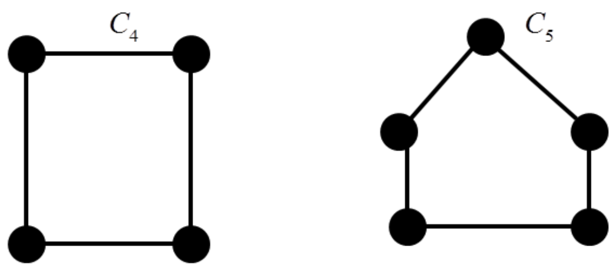

Figure 2.5: Cyclic Graph

Cyclic Graphs

In a cyclic graph, vertices are arranged in a ring format as shown in the Fig. 2.5. The vertex and edge sets of a cyclic graph are as follows

$$V = \{\nu_1, \nu_2, ...\nu_n\}$$

$$E = \{(\nu_1, \nu_2), (\nu_2, \nu_3), ... (\nu_j, \nu_{j+1}), ... (\nu_{n-1}, \nu_n), (\nu_n, \nu_1)\}$$

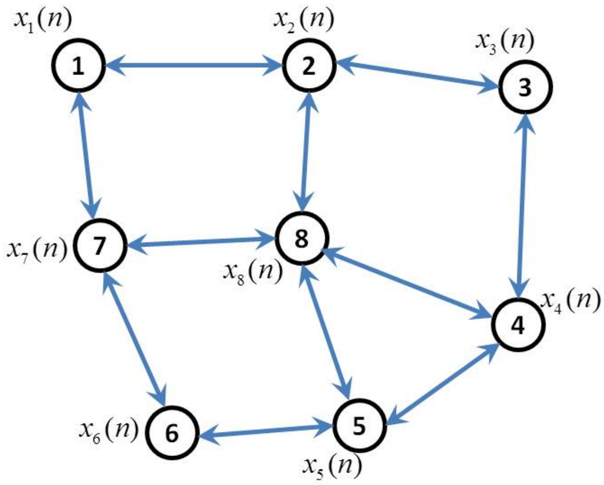

Figure 2.6: Distributed Consensus Algorithm for Wireless Sensor Network

2.2 Consensus Algorithms

The consensus is a process to reach an agreement regarding a certain quantity of interest that depends on the state of all the nodes. A consensus algorithm is an interaction rule that specifies the information exchange between a node and all of its direct neighbors [6]. These algorithms have received a lot of attention due to their ability to compute the desired global statistics by exchanging information only with the direct neighbors. Average consensus algorithms have been extensively studied in distributed agreement and synchronization problems in the multi-agent systems and load balancing in parallel computers [28]. They can also be utilized in the applications of mobile autonomous agents [6], [9], [10] and distributed data fusion in sensor networks [15], [29].

Average Consensus algorithm

Let $x_i(0)$ be the real scalar assigned to the node i at $t = 0$. Average consensus algorithm computes the average $x_{avg} = \frac{\sum_{i=1}^{n} x_i(0)}{n}$ at every node through a decentralized approach which does not require the sink node/base station as shown in Fig. 2.6. At each step, node *'i'* carries out its update based on its local state and communication with it's direct neighbors.

$$x_i(t+1) = x_i(t) + h \sum_{j \in N_i} (x_j(t) - x_i(t)), \quad i = 1, ..., n, \tag{2.2.1}$$

where h is a consensus parameter and N_i denotes neighbor set of node i. This iterative method is expressed as the simple linear iteration

$$x(t+1) = Wx(t), \quad t = 0, 1, 2..., \tag{2.2.2}$$

where W denotes the weight matrix, and W_{ij} is the weight associated with the edge (i, j). If we assign equal weight h to each link in the network, then from [30], optimal weight for a given topology is

$$W_{ij} = \begin{cases} h & if, \quad (i,j) \in E, \\ 1 - hdeg(\nu_i) & if, \quad i = j, \\ 0 & otherwise. \end{cases} \tag{2.2.3}$$

The optimal link weight or the consensus parameter [30] is defined as

$$h = \frac{2}{\lambda_2(L) + \lambda_n(L)}. \tag{2.2.4}$$

and weight matrix W is given by

$$W = I - hL. \tag{2.2.5}$$

where I is an $n \times n$ identity matrix. Let $\lambda_n(W)$ be the n^{th} eigenvalue of W, then $\lambda_n(W) = 1 - h\lambda_n(L)$ satisfies

$$1 = \lambda_1(W) > \lambda_2(W) > \lambda_3(W).........\lambda_n(W). \tag{2.2.6}$$

The weight matrix W satisfies

$$W = W^T, W\mathbf{1} = \mathbf{1}, W \in S$$

where $\mathbf{1}$ denotes the $[1, 1, ...1]^T$ and S denotes the matrices

$$S = \left\{W \in R^{n \times n} \left| W_{ij} = 0 \; if \; i \neq j \; and \; \{i, j\} \notin \varepsilon \right.\right\}$$

Node values to achieve asymptotic average consensus is expressed as

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} W^t x(0) = \frac{\mathbf{1}\mathbf{1}^T x(0)}{n}$$

Thus, the weight matrix should satisfy the following condition

$$\lim_{t \to \infty} W^t = \frac{11^T}{n}.$$

It can be also written as

$$\|W - J\| < 1$$

Convergence rate of consensus algorithms is defined as the speed at which every node converges to the average of the intial state values. To evaluate the convergence rate, there are many algorithms available in the literature, such as *best constant* weights algorithm, *metropolis-hastings* weights algorithm, *max-degree* weights algorithm [7].

2.3 Best Constant Weights Algorithm

Best constant weights algorithm gives the fastest convergence rate among the other uniform weight methods [12], [23]. In this thesis, we use this algorithm to derive the closed-form expressions of convergence rate for regular graph models.

(1) Observe the second largest and smallest eigenvalues of W.

(2) Calculate h using $|\lambda_2(W)| = |\lambda_n(W)|$, here $\lambda_2(W)$ is a second largest eigenvalue of W and $\lambda_n(W)$ is a smallest eigenvalue of W.

(3) Determine $\gamma(W) = max \{|1 - h\lambda_2(L)|, |1 - h\lambda_n(L)|\}$.

(4) Calculate the convergence rate [60]

$$R = 1 - \gamma. \tag{2.3.1}$$

Convergence time [7] of a consensus algorithm is defined as the number of steps to reach the global average. Convergence time of a consensus algorithm is measured by

$$T = \frac{1}{log(\frac{1}{\gamma})}. \tag{2.3.2}$$

2.4 Metropolis-Hastings Weights Algorithm

In *metropolis-hastings* weights algorithm, weights between node i and j can be expressed as

$$W_{ij} = \begin{cases} \frac{1}{(d+1)} & i \neq j, \qquad \{i,j\} \in \varepsilon \\ 1 - \frac{d_i}{d+1} & i = j \\ 0 & i \neq j, \{i,j\} \in \varepsilon \end{cases}$$

where d_i is the degree of the node i and $d = max(d_i)$ is the degree of the graph. These weights can be called as max-degree weights.

2.5 Max-Degree Weights Algorithm

In *max-degree* weights algorithm, weights between node i and j can be expressed as

$$W_{ij} = \begin{cases} \frac{1}{max\{d_i,d_j\}+1} & i \neq j, \quad \{i,j\} \in \varepsilon \\ 1 - \sum\limits_{j \in N_i} \frac{1}{\max\{d_i,d_j\}+1} & i = j \\ 0 & i \neq j, \quad \{i,j\} \notin \varepsilon \end{cases}$$

2.6 Gossip Algorithms

Gossip algorithm is an asynchronous consensus algorithm, where the node pairs interact and update with the average of their previous state values in every iteration. It is a distributed operation which enables the sensor nodes to asymptotically determine the average of their initial gossip variables. Gossip algorithms [31], [32], [33], [34], [35], [36], [37], [38], [39] have generated a lot of attention in the last decade due to their ability to achieve the global average using pairwise communications between nodes. They are quite suitable for data delivery in WSNs [21], [39] as they can be utilized when the global network topology is highly dynamic, and the network consists of power constrained nodes. In this algorithm, nodes do not broadcast the information to their neighbors. To achieve the faster convergence rates, all disjoint pairs gossip at every time instant considering the periodic gossip sequences. These algorithms are called periodic gossip algorithms [36], [37]. In every step, periodic gossip sequences enable the node pairs to participate simultaneously in the gossip process. If the period of a periodic gossip sequence is equal to the chromatic index of the graph, then it is called as an optimal periodic gossip sequence. An optimal periodic sequence ensures the faster convergence rates using a minimum number of steps. As the gossip algorithms are iterative, the convergence rate of the algorithms greatly influences the performance of the WSNs. Convergence rate of a periodic gossip algorithm is characterized by the magnitude of the second largest eigenvalue of a gossip matrix [35].

Periodic Gossip Algorithms

The gossiping process can be modeled as a discrete time linear system [33] as

$$\mathbf{x}(t+1) = M(t)\mathbf{x}(t), \quad t = 1, 2, .. \tag{2.6.1}$$

where $\mathbf{x}$ is a vector of node variables, and $M(t)$ denotes a doubly stochastic matrix. If nodes i and j gossip at time t, then the values of nodes at time $t+1$ will be updated as

$$x_i(t+1) = x_j(t+1) = \frac{x_i(t) + x_j(t)}{2}. \tag{2.6.2}$$

$M(t)$ is expressed as $M(t)$=$P_{i,j}$, where P_{ij}=$[P_{lm}]_{n\times n}$ for each step (i,j) with entries defined as

$$p_{lm} = \begin{cases} \frac{1}{2}, & (l,m) \in (i,i),(i,j),(j,i),(j,j) \\ 1, & l = m, l \neq i, l \neq j; \\ 0, & otherwise. \end{cases} \tag{2.6.3}$$

A gossip sequence is defined as an sequence of edges for a given network in which each pair appears exactly once in one time step. For a gossip sequence $(i_1, j_1), (i_2, j_2), ...(i_k, j_k)$, the gossip matrix is expressed as $P_{i_k j_k}.......P_{i_2 j_2}P_{i_1 j_1}$. For a periodic gossip sequence with period T, if i_t, j_t denotes t^{th} gossip pair, then $i_{T+k} = i_k$ for $k = 1, 2, ...$

Here, we can write variable x at $(k+1)$ as

$$\mathbf{x((k+1)T)} = W\mathbf{x(kT)}, k = 0, 1, 2...n,$$

where W is a doubly stochastic matrix. T also denotes the number of steps needed to implement its one-period sub-sequence E.

The subset of edges is such that no two edges are adjacent to the same node and the gossips on these edges can be performed simultaneously in one-time step is defined as multi-gossip. The minimum value of T is defined by the chromatic index. The value of the chromatic index is either d_{max} or $d_{max} + 1$, where d_{max} is the maximum degree of a graph. When multi-gossip is allowed, a periodic gossip sequence E, E, E with T=chromatic index is called an optimal periodic gossip sequence [35].

Convergence rate

Convergence rate [33], [35] of the periodic gossip algorithms is characterized by the second largest eigenvalue of weight matrix. Convergence rate (R) at which gossip variable converges to a rank one matrix is determined by the spectral gap [31], [32]

$$R = 1 - \left|\lambda_2(W)\right|. \tag{2.6.4}$$

Convergence time of a gossip algorithm is defined as the number of steps to reach the global average. It is measured by [40]

$$T = \frac{1}{log(\frac{1}{\gamma})}. \tag{2.6.5}$$

Chapter 3

Analysis of Average Consensus Algorithm for Wireless Sensor Networks

3.1 Introduction

Distributed average consensus algorithms can be applied to WSNs for data fusion [15], [21], [22], [23], [29], [41], [42], [43]. As the consensus algorithms are iterative in nature, the convergence rate of the algorithms greatly influences the performance of the WSNs, and it is lower bounded by the second smallest eigenvalue of the graph Laplacian [27]. To make this algorithm useful to WSN scenarios, it is necessary to study the convergence rate for large-scale networks. In [15], authors studied the convergence of the consensus algorithm for WSNs using random topologies. In [22], authors proposed the approach for consensus and derived the performance bounds in terms of eigenvalues. In [23], authors proposed a new average consensus algorithm and shown that their algorithm shows faster convergence rate for realistic topologies over existing consensus protocols. A distributed iterative algorithm based on average consensus has been proposed in [29], to compute the maximum-likelihood estimate of the parameters. In [17], the distributed average consensus has been considered when the topology is random, and the communication in the channels is corrupted by additive noise. It was proved that running the consensus for a long time reduces the bias of the final average estimate but increases its variance. A closed-form expression for the mean square error of the state and the optimum choice of parameters have been derived in [43] to guarantee the fastest convergence. Consensus on small world and Ramanujan networks has been studied in [44], [45] and it has been proved that the convergence rate is maximized for these topologies. Optimal topology framework which increases the convergence rate and minimizes the energy consumption has been studied in [46]. In our work, we study the convergence of the consensus algorithm for finite distance-regular networks with the varying number of nearest neighbors[1]. These finite sized networks represent the notion of geographical proximity in the practical WSNs. The main motivation for using the regular graph model is that most of the practical WSNs are finite sized which cannot be studied by asymptotic results existing in the literature. In r-nearest neighbor ring and torus network, an edge exists between every pair of neighbors that are *'r'* hops away. If a node's transmission radius is increased, it will able to communicate with more number of nodes. So, the variable *'r'* can model the transmission radius and in WSNs. Most of the WSN applications, such as space monitoring, cave monitoring, and underwater ecosystems operate in multiple dimensions. So without loss of generality, we have also derived the expressions for closed-form expressions of convergence rate for m-dimensional r-nearest neighbor torus networks. Distributed average consensus algorithms are simple to implement for WSNs. But, it is generally difficult to predict its convergence rate for large-scale networks. Although there have been several studies in the literature, analytic tools to control the network performance for large-scale

[1]**Sateeshkrishna Dhuli**, Kumar Gaurav and Y. N. Singh, "Convergence analysis for regular wireless consensus networks," *IEEE Sensors Journal* , vol. 15, no. 8, pp. 4522-4531, Aug 2015.

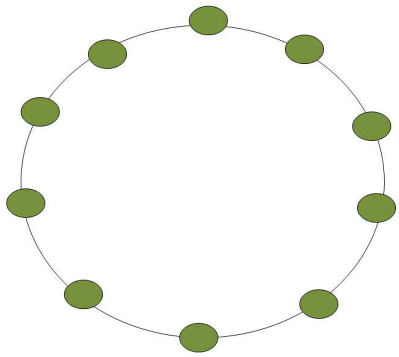

Figure 3.1: 1-nearest neighbor ring network.

WSNs are still inadequate. In this chapter, we derive the generalized expressions of convergence rate for large-scale WSNs. This kind of analysis helps in estimating the convergence rate efficiently as it avoids usage of computationally expensive algorithms which depends on thousands of simulation trials.

Organization

This chapter is organized as follows. In section 3.2, we derive the closed-form expressions of convergence rate of average consensus algorithm for r-nearest neighbor ring network. In section 3.3, we derive the closed-form expressions of convergence rate for r-nearest neighbor torus networks. In section 3.4, we provide the convergence rate expressions for m-dimensional r-nearest neighbor networks. Numerical results have been presented in section 3.5. Finally, we discuss the conclusions of this chapter in section 3.6.

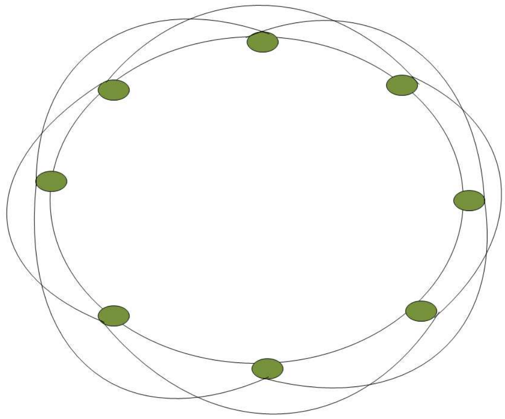

Figure 3.2: 2-nearest neighbor ring network.

3.2 r-Nearest Neighbor Ring Networks

In this section, we derive the convergence rate of average consensus algorithms for r-nearest neighbor network. A ring network or 1-nearest neighbor ring network is as shown in the Fig. 3.1. In a 2-nearest neighbor ring network, nodes are connected to direct neighbors and also to the nodes which are 2-hops away as shown in the Fig. 3.2. In a r-nearest neighbor ring network, nodes are connected to the nodes which are r-hops away. Ring network can be represented by a circulant matrix as

$$\begin{bmatrix} a_1 & a_2 & \ldots\ldots & a_{n-1} & a_n \\ a_n & a_1 & \ldots\ldots & a_{n-2} & a_{n-1} \\ . & . & & . & . \\ . & . & & . & . \\ a_3 & a_4 & \ldots\ldots & a_1 & a_2 \\ a_2 & a_3 & \ldots\ldots & a_n & a_1 \end{bmatrix}. \tag{3.2.1}$$

where a_t, $t = 1$ to n represents the topology coefficients.

Definition 3.2.1. *The $(k+1)^{th}$ eigenvalue [47] of a circulant matrix $circ(a_1, a_2......a_n)$ is defined as*

$$\lambda_k = a_1 + a_2\omega^k + ... + a_n\omega^{(n-1)k}, k = 0, 1....n-1 \tag{3.2.2}$$

where ω is the n^{th} root of 1, given by $\omega = e^{\frac{2\pi i}{n}}$.

The ring network and 2-nearest neighbor ring network are shown in Fig. 3.1 and Fig. 3.2 respectively. Then adjacency matrix (A) of a ring network is

$$A = \begin{bmatrix} 0\ 1\ 0 \ldots\ldots\ldots\ldots 0\ 1 \\ 1\ 0\ 1 \ldots\ldots\ldots\ldots 0\ 0 \\ .\ \ .\ \ . \qquad\qquad\quad .\ \ . \\ .\ \ .\ \ . \qquad\qquad\quad .\ \ . \\ 0\ 0\ 0 \ldots\ldots\ldots\ldots 0\ 1 \\ 1\ 0\ 0 \ldots\ldots\ldots\ldots 1\ 0 \end{bmatrix}, \tag{3.2.3}$$

and degree matrix (D) of a ring network is expressed as

$$D = \begin{bmatrix} 2\ 0\ 0 \ldots\ldots\ldots\ldots 0\ 0 \\ 0\ 2\ 0 \ldots\ldots\ldots\ldots 0\ 0 \\ .\ \ . \qquad\qquad\quad .\ \ . \\ .\ \ . \qquad\qquad\quad .\ \ . \\ 0\ 0 \ldots\ldots\ldots\ldots 2\ 0 \\ 0\ 0 \ldots\ldots\ldots\ldots 0\ 2 \end{bmatrix}. \tag{3.2.4}$$

Lemma 3.2.1. *The $(k+1)^{th}$ eigenvalue of a weight matrix (W) for ring network [28] is*

$$\lambda_k(W) = (1-2h) + 2h\cos\left(\tfrac{2\pi k}{n}\right), \tag{3.2.5}$$

where $k = 0, 1, ...(n-1)$.

Proof. Using (3.2.3) and (3.2.4), Laplacian matrix for a ring network is expressed as

$$L = \begin{bmatrix} 2 & -1 & 0 \ldots\ldots\ldots\ldots 0 & -1 \\ -1 & 2 & -1 \ldots\ldots\ldots\ldots 0 & 0 \\ . & . & . \qquad\qquad . & . \\ . & . & . \qquad\qquad . & . \\ 0 & 0 & 0 \ldots\ldots\ldots\ldots 2 & -1 \\ -1 & 0 & 0 \ldots\ldots\ldots -1 & 2 \end{bmatrix}. \tag{3.2.6}$$

Using (2.2.5) and (3.2.6), we get

$$W = (I - Lh) = \begin{bmatrix} (1-2h) & h & 0 \text{ ....}0 & h \\ h & (1-2h) & h \text{ ....}0 & 0 \\ . & . & . \qquad\qquad . & . \\ . & . & . \qquad\qquad . & . \\ 0 & 0 & 0 \text{ ....}(1-2h) & h \\ h & 0 & 0 \text{ ....}h & (1-2h) \end{bmatrix}. \quad (3.2.7)$$

(3.2.7) is a circulant matrix. Hence $(k+1)^{th}$ eigenvalue of W is simplified as (3.2.5). ■

Theorem 3.2.1. *The $(k+1)^{th}$ eigenvalue of a weight matrix (W) for r-nearest neighbor ring network is*

$$\lambda_k(W) = (1 - 2rh) + 2h \sum_{j=1}^{r} \cos\left(\tfrac{2\pi kj}{n}\right), \quad (3.2.8)$$

where $k = 0, 1, ...(n-1)$.

Proof. The Adjacency matrix (A) of a r-nearest neighbor ring network is

$$A = circ(0, \underbrace{1, 1, .., 1}_{r\ terms}, \underbrace{0, 0, .., 0}_{n-2r-1\ terms}, \underbrace{1, 1, .., 1}_{r\ terms}), \quad (3.2.9)$$

Similarly, degree matrix (D) of a r-nearest neighbor ring network is

$$D = circ(2r, 0, 0, 0, ...0), \quad (3.2.10)$$

From (3.2.9) and (3.2.10), the Laplacian matrix (L) is expressed as

$$L = circ(2r, \underbrace{-1, -1, .., -1}_{r\ terms}, \underbrace{0, 0, ..., 0}_{n-2r-1\ terms}, \underbrace{-1, -1, ..., -1}_{r\ terms}), \quad (3.2.11)$$

Using (3.2.11) and (2.2.5), weight matrix W for r-nearest neighbor ring network is expressed as

$$W = circ(1 - 2rh, \underbrace{h, h, ..., h}_{r\ terms}, \underbrace{0, 0, ..., 0}_{n-2r-1\ terms}, \underbrace{h, h, ..., h}_{r\ terms}), \quad (3.2.12)$$

Applying (3.2.2) on (3.2.12) gives the $(k+1)^{th}$ eigenvalue of W for r-nearest neighbor ring network as (3.2.8). ■

Theorem 3.2.2. *The convergence rate R of an r-nearest neighbor ring network C_n^r for even number of nodes is expressed as*

$$R = 1 - \left| \frac{\left(\frac{\sin\left(\frac{(2r+1)\pi}{n}\right)}{\sin\frac{\pi}{n}} - \cos(\pi r) \right)}{4r + 2 - \left(\frac{\sin\left(\frac{(2r+1)\pi}{n}\right)}{\sin\frac{\pi}{n}} + \cos(\pi r) \right)} \right|. \tag{3.2.13}$$

Proof. From (3.2.8), second largest eigenvalue of weight matrix for even number of nodes can be written as

$$\lambda_1(W) = (1 - 2rh) + 2h \sum_{j=1}^{r} \cos\left(\frac{2\pi j}{n}\right), \tag{3.2.14}$$

Similarly, smallest eigenvalue of a weight matrix for even number of nodes can be written as

$$\lambda_{\frac{n}{2}}(W) = (1 - 2rh) + 2h \sum_{j=1}^{r} \cos(\pi j), \tag{3.2.15}$$

From best-constant algorithm, γ is minimum, when

$$|\lambda_1(W)| = \left|\lambda_{\frac{n}{2}}(W)\right|. \tag{3.2.16}$$

Substitution of (3.2.14) and (3.2.15) in (3.2.16), results in

$$h = \frac{1}{2r + 0.5 - \sum\limits_{j=1}^{r} \cos\left(\frac{2\pi j}{n}\right) - \sum\limits_{j=1}^{r} \cos(\pi j)}. \tag{3.2.17}$$

Definition 3.2.2. *Dirichlet kernel is defined as*

$$1 + 2\sum_{j=1}^{r} \cos(jx) = \frac{\sin\left(r + \frac{1}{2}\right)x}{\sin\frac{x}{2}}. \tag{3.2.18}$$

Using (3.2.18), h can be rewritten as

$$h = \frac{1}{2r+1-\frac{1}{2}\left(\frac{\sin\left(\frac{(2r+1)\pi}{n}\right)}{\sin\frac{\pi}{n}} - \frac{\cos\left(\frac{\pi(2r+1)}{2n}\right)}{\cos\frac{\pi}{2n}}\right)}. \tag{3.2.19}$$

Finally, substitution of (3.2.19) in (3.2.14) gives γ as

$$\gamma = \frac{\left(\frac{\sin\left(\frac{(2r+1)\pi}{n}\right)}{\sin\frac{\pi}{n}} - \cos(\pi r)\right)}{4r+2-\left(\frac{\sin\left(\frac{(2r+1)\pi}{n}\right)}{\sin\frac{\pi}{n}} + \cos(\pi r)\right)}. \tag{3.2.20}$$

Substituting (3.2.20) in (2.3.1) proves the Theorem. ■

Theorem 3.2.3. *Convergence rate of a r-nearest neighbor ring network C_n^r for $n = odd$ is expressed as*

$$R = 1 - \left|\frac{\left(\frac{\sin\left(\frac{(2r+1)\pi}{n}\right)}{\sin\frac{\pi}{n}} + \frac{\cos\left(\frac{\pi(2r+1)}{2n}\right)}{\cos\frac{\pi}{2n}}\right)}{4r+2-\left(\frac{\sin\left(\frac{(2r+1)\pi}{n}\right)}{\sin\frac{\pi}{n}} - \frac{\cos\left(\frac{\pi(2r+1)}{2n}\right)}{\cos\frac{\pi}{2n}}\right)}\right|. \tag{3.2.21}$$

Proof. From (3.2.8), second largest eigenvalue of weight matrix for n=odd can be written as

$$\lambda_1(W) = (1-2rh) + 2h\sum_{j=1}^{r}\cos\left(\frac{2\pi j}{n}\right), \tag{3.2.22}$$

Similarly, smallest eigenvalue of a weight matrix for n=odd can be written as

$$\lambda_{\frac{(n-1)}{2}}(W) = (1-2hr) + 2h\sum_{j=1}^{r}\cos\left(\frac{\pi j(n-1)}{n}\right), \tag{3.2.23}$$

γ is minimum, when

$$|\lambda_1(W)| = \left|\lambda_{\frac{(n-1)}{2}}(W)\right|. \tag{3.2.24}$$

Substitution of (3.2.22) and (3.2.23) in (3.2.24), results in

$$h = \frac{1}{2r - \sum_{j=1}^{r} \cos\left(\frac{2\pi j}{n}\right) - \sum_{j=1}^{r} \cos\left(\frac{\pi j(n-2)}{n}\right)}, \tag{3.2.25}$$

By using (3.2.18), we can rewrite the h as

$$h = \frac{1}{2r+1-\frac{1}{2}\left(\frac{\sin\left(\frac{(2r+1)\pi}{n}\right)}{\sin\frac{\pi}{n}} - \frac{\cos\left(\frac{\pi(2r+1)}{2n}\right)}{\cos\frac{\pi}{2n}}\right)}. \tag{3.2.26}$$

Substituting the (3.2.26) in (3.2.22) results in

$$\gamma = \frac{\left(\frac{\sin\left(\frac{(2r+1)\pi}{n}\right)}{\sin\frac{\pi}{n}} + \frac{\cos\left(\frac{\pi(2r+1)}{2n}\right)}{\cos\frac{\pi}{2n}}\right)}{4r+2-\left(\frac{\sin\left(\frac{(2r+1)\pi}{n}\right)}{\sin\frac{\pi}{n}} - \frac{\cos\left(\frac{\pi(2r+1)}{2n}\right)}{\cos\frac{\pi}{2n}}\right)}. \tag{3.2.27}$$

Substituting the (3.2.27) in (2.6.1) proves the theorem. ■

3.3 r-Nearest Neighbor Torus Network

A torus network or 1-nearest neighbor torus is as shown in the Fig. 3.3. In a 2-nearest neighbor torus network, nodes are connected to direct neighbors and also to the nodes which are 2-hops away as shown in the Fig. 3.4. Similarly, nodes are connected to direct neighbors and also to the nodes which are r-hops away in a r-nearest neighbor torus network. A torus network can be represented by $n \times n$ block circulant matrix A as

$$A = \begin{bmatrix} A_0 & A_1 \ldots\ldots\ldots A_{n_1-2} & A_{n_1-1} \\ A_{n_1-1} & A_0 \ldots\ldots\ldots A_{n_1-3} & A_{n_1-2} \\ . & . \quad\quad . & . \\ . & . \quad\quad . & . \\ A_1 & A_2 \ldots\ldots\ldots A_{n_1-1} & A_0 \end{bmatrix}, \tag{3.3.1}$$

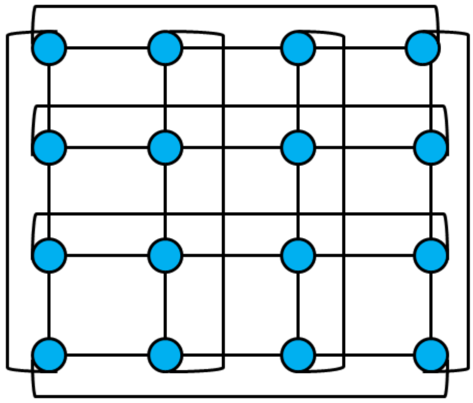

Figure 3.3: Torus Network

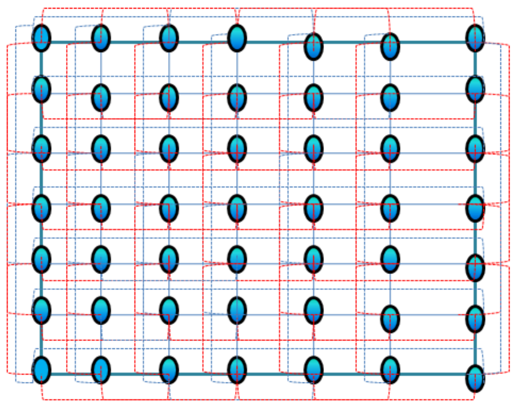

Figure 3.4: 2-Nearest Neighbor Torus Network

Let the number of nodes $n = n_1^2$, then each block A_i, for $i = 0, 1...(n_1 - 1)$ represents $n_1 \times n_1$ circulant matrices.

Lemma 3.3.1. *The eigenvalue* λ_{j_1,j_2} *of* W_{k_1,k_2} *of* 1*-nearest neighbor torus [28] is*

$$\lambda_{j_1,j_2}(W_{k_1,k_2}) = 1 - 4h + 2h\cos\left(\frac{2\pi j_1}{k_1}\right) + 2h\cos\left(\frac{2\pi j_2}{k_2}\right), \tag{3.3.2}$$

where $j_1 = 0, 1, 2, ...(k_1 - 1)$, $j_2 = 0, 1, 2, ...(k_2 - 1)$.

Proof. A torus network T_{k_1,k_2} is a result of the Cartesian product of two ring networks C_{k_1} and C_{k_2} with k_1 and k_2 nodes respectively. The eigenvalue expression for Laplacian of a torus network is the addition of corresponding eigenvalues of Laplacian of two circulant networks with k_1 and k_2 nodes [28]. The $(j_1 + 1)^{th}$ eigenvalue of a Laplacian matrix (L) for a ring network is

$$\lambda_{j_1}(L) = 2 - 2\cos\frac{2\pi j_1}{k_1}, \tag{3.3.3}$$

The $(j_2 + 1)^{th}$ eigenvalue of a Laplacian matrix (L) for a ring network is

$$\lambda_{j_2}(L) = 2 - 2\cos\frac{2\pi j_2}{k_2}, \tag{3.3.4}$$

Using (3.3.3) and (3.3.4), we can write the eigenvalue of a Laplacian matrix for a torus network is

$$\lambda_{j_1,j_2}(L) = 4 - 2\cos\frac{2\pi j_1}{k_1} - 2\cos\frac{2\pi j_2}{k_2} \tag{3.3.5}$$

From (3.3.5), (2.2.5), we can write the eigenvalue of a W_{k_1,k_2} for a torus network as (3.3.2). ∎

Theorem 3.3.1. *The eigenvalue* λ_{j_1,j_2} *of* W_{k_1,k_2} *for* r*-nearest neighbor torus is*

$$\lambda_{j_1,j_2}(W_{k_1,k_2}) = (1 - 4rh) + 2h\sum_{j=1}^{r}\cos\left(\frac{2\pi j_1 j}{k_1}\right) + 2h\sum_{j=1}^{r}\cos\left(\frac{2\pi j_2 j}{k_2}\right), \tag{3.3.6}$$

where $j_1 = 0, 1, 2, ...(k_1 - 1), j_2 = 0, 1, 2, ...(k_2 - 1)$.

Proof. The weight matrix for r-nearest neighbor torus is

$$W = circ(W_{k_1} - 2rhI_{k_1}, \underbrace{hI_{k_1}, hI_{k_1}, ...hI_{k_1}}_{r\ terms}, ... \underbrace{0, 0, 0, 0, ...0}_{k_2-2r-1\ terms} .. \underbrace{hI_{k_1}, hI_{k_1}, ...hI_{k_1}}_{r\ terms}), \quad (3.3.7)$$

From (3.2.2) and (3.3.7), we obtain

$$\lambda_{j_1,j_2}(W_{k_1,k_2}) = \lambda\left(W_{k_1}\right) - 2rh + 2rh \sum_{j=1}^{r} \cos\left(\frac{2\pi j j_2}{k_2}\right), \quad (3.3.8)$$

The $(j_1+1)^{th}$ eigenvalue of a weight matrix (W_{k_1}) is

$$\lambda_{j_1}(W_{k_1}) = (1 - 2rh) + 2h \sum_{j=1}^{r} \cos\left(\frac{2\pi j_1 j}{k_1}\right), \quad (3.3.9)$$

Therefore, substitution of (3.3.9) in (3.3.8) results in (3.3.6). ■

Theorem 3.3.2. *Convergence rate of an r-nearest neighbor torus $T^r_{k_1,k_2}$ for $k_1 = k_2 =$ even, is expressed as*

$$R = 1 - \left| \frac{r + 0.5 + 0.5\left(\frac{\sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} - 2\cos\pi r\right)}{(1.5 + 3r) - 0.5\left(\frac{\sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} + 2\cos\pi r\right)} \right|. \quad (3.3.10)$$

Proof. Using (3.3.6), we can write the second largest eigenvalue of a weight matrix for $k_1 = k_2 = even$ can be written as

$$(1 - 2hr) + 2h \sum_{j=1}^{r} \cos\left(\frac{2\pi j}{k_{max}}\right), \quad (3.3.11)$$

where $k_{max} = max(k_1, k_2)$.

Using (3.3.6), the smallest eigenvalue of a weight matrix for $k_1 = k_2 = even$ can be written as

$$\lambda_{\frac{k_1}{2},\frac{k_2}{2}}(W) = (1 - 4hr) + 4h \sum_{j=1}^{r} \cos\left(\pi j\right), \quad (3.3.12)$$

γ is minimum, when

$$SLEM = \left|\lambda_{\frac{k_1}{2},\frac{k_2}{2}}(W)\right|. \tag{3.3.13}$$

Here, SLEM is abbreviated as second largest eigenvalue modulus.

Substitution of (3.3.11) and (3.3.12) in (3.3.13) results in

$$h = \frac{1}{3r - \sum_{i=1}^{r} \cos\left(\frac{2\pi i}{k_{max}}\right) - 2\sum_{i=1}^{r} \cos(\pi i)}, \tag{3.3.14}$$

Substitution of (3.3.14) in (3.3.11) results in

$$\gamma = \frac{r + 0.5 + 0.5\left(\frac{\sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} - 2\cos \pi r\right)}{(1.5 + 3r) - 0.5\left(\frac{\sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} + 2\cos \pi r\right)}. \tag{3.3.15}$$

Finally, we can prove the theorem using (3.3.15) and (2.3.1). ■

Theorem 3.3.3. *Convergence rate of an r-nearest neighbor torus $T^r_{k_1,k_2}$ for $k_1 = k_2 =$ odd, is expressed as*

$$R = 1 - \left|\frac{r+0.5+0.5\left(\frac{\sin\left(\frac{\pi(2r+1)(k_1-1)}{2k_1}\right)}{\sin\left(\frac{\pi(k_1-1)}{2k_1}\right)} - \frac{\sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} + \frac{\sin\left(\frac{\pi(2r+1)(k_2-1)}{2k_2}\right)}{\sin\left(\frac{\pi(k_2-1)}{2k_2}\right)}\right)}{(1.5+3r)-0.5\left(\frac{\sin\left(\frac{\pi(2r+1)(k_1-1)}{2k_1}\right)}{\sin\left(\frac{\pi(k_1-1)}{2k_1}\right)} + \frac{\sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} + \frac{\sin\left(\frac{\pi(2r+1)(k_2-1)}{2k_2}\right)}{\sin\left(\frac{\pi(k_2-1)}{2k_2}\right)}\right)}\right|. \tag{3.3.16}$$

Proof. Using (3.3.6), we can write the second largest eigenvalue of a weight matrix for $k_1 = k_2 = odd$ as

$$(1 - 2hr) + 2h\sum_{j=1}^{r} \cos\left(\frac{2\pi j}{k_{max}}\right), \tag{3.3.17}$$

where $k_{max} = max(k_1, k_2)$.

Similarly, smallest eigenvalue of a weight matrix for $k_1 = k_2 = odd$ as

$$\lambda_{\frac{(k_1-1)}{2},\frac{(k_2-1)}{2}}(W) = (1-4hr) + 2h\sum_{j=1}^{r}\cos\left(\frac{\pi j(k_1-1)}{k_1}\right) + 2h\sum_{j=1}^{r}\cos\left(\frac{\pi j(k_2-1)}{k_2}\right). \quad (3.3.18)$$

Convergence factor γ is minimum when,

$$SLEM = \left|\lambda_{\frac{(k_1-1)}{2},\frac{(k_2-1)}{2}}(W)\right|. \quad (3.3.19)$$

Substitution of (3.3.17) and (3.3.18) in (3.3.19) results in

$$h = \frac{1}{3r - \sum_{i=1}^{r}\cos\left(\frac{2\pi i}{k_2}\right) - \sum_{i=1}^{r}\cos\left(\frac{\pi i(k_1-1)}{k_1}\right) - \sum_{i=1}^{r}\cos\left(\frac{\pi i(k_2-1)}{k_2}\right)}. \quad (3.3.20)$$

Using (3.2.18), h can be further simplified as

$$h = \frac{1}{1.5+3r-0.5\left(\frac{\sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} + \frac{\sin\left(\frac{\pi(2r+1)(k_1-1)}{2k_1}\right)}{\sin\left(\frac{\pi(k_1-1)}{2k_1}\right)} + \frac{\sin\left(\frac{\pi(2r+1)(k_2-1)}{2k_2}\right)}{\sin\left(\frac{\pi(k_2-1)}{2k_2}\right)}\right)}. \quad (3.3.21)$$

Substitution of (3.3.21) in (3.3.17) results in γ as

$$\gamma = \frac{r+0.5+0.5\left(\frac{\sin\left(\frac{\pi(2r+1)(k_1-1)}{2k_1}\right)}{\sin\left(\frac{\pi(k_1-1)}{2k_1}\right)} - \frac{\sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} + \frac{\sin\left(\frac{\pi(2r+1)(k_2-1)}{2k_2}\right)}{\sin\left(\frac{\pi(k_2-1)}{2k_2}\right)}\right)}{(1.5+3r)-0.5\left(\frac{\sin\left(\frac{\pi(2r+1)(k_1-1)}{2k_1}\right)}{\sin\left(\frac{\pi(k_1-1)}{2k_1}\right)} + \frac{\sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} + \frac{\sin\left(\frac{\pi(2r+1)(k_2-1)}{2k_2}\right)}{\sin\left(\frac{\pi(k_2-1)}{2k_2}\right)}\right)}. \quad (3.3.22)$$

Finally, substitution of γ in (2.3.1) proves the theorem. ■

3.4 m-Dimensional r-Nearest Neighbor Torus Network

In some WSN applications, nodes need to operate in more than two dimensions such as under water WSNs, WSNs on remote hilly areas. This section investigates the effect of network dimension on convergence rate for average consensus algorithms.

Theorem 3.4.1. *The eigenvalues $\lambda_{j_1,j_2,....j_m}$ of weight matrix $W_{k_1,k_2,....k_m}$ for m-dimensional r-nearest neighbor torus is*

$$\lambda_{j_1,j_2,....j_m}(W) = (1 - 2mrh) + 2h \sum_{j=1}^{r} \sum_{l=1}^{m} \cos\left(\frac{2\pi j_l}{k_l}\right), \tag{3.4.1}$$

where $j_l = 0, 1, 2, ...(k_l - 1)$.

Proof. From (3.3.5), the eigenvalue expression for two dimensional r-nearest neighbor torus is

$$\lambda_{j_1,j_2}(W_{k_1,k_2}) = (1 - 4rh) + 2h \sum_{l=1}^{r} \cos\left(\tfrac{2\pi j_1 l}{k_1}\right) + 2h \sum_{l=1}^{r} \cos\left(\tfrac{2\pi j_2 l}{k_2}\right), \tag{3.4.2}$$

Similarly, eigenvalue expression for three dimensional r-nearest neighbor torus can be expressed as

$$\lambda_{j_1,j_2,j_3}(W_{k_1,k_2,k_3}) = (1 - 6rh) + 2rh \sum_{l=1}^{r} \cos\left(\tfrac{2\pi j_1 l}{k_1}\right) + 2rh \sum_{l=1}^{r} \cos\left(\tfrac{2\pi j_2 l}{k_2}\right) + 2rh \sum_{l=1}^{r} \cos\left(\tfrac{2\pi j_3 l}{k_3}\right), \tag{3.4.3}$$

Hence, without loss of generality, from (3.4.2) and (3.4.3), the eigenvalue expression of m-dimensional r-nearest neighbor torus can be written as (3.4.1). ■

Theorem 3.4.2. *Convergence rate of a m-dimensional r-nearest neighbor torus for $k_1 = k_2 = ... = k_m = even$ is expressed as*

$$R = 1 - \left| \frac{(m-1)(r+0.5) + \frac{0.5\sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} - \frac{m}{2}\cos(\pi r)}{(m+1)(r+0.5) - \frac{0.5\sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} - \frac{m}{2}\cos(\pi r)} \right|. \tag{3.4.4}$$

Proof. From (3.4.1), second largest eigenvalue of a weight matrix can be written as

$$(1 - mhr) + 2h \sum_{j=1}^{r} \cos\left(\frac{2\pi j}{k_{max}}\right), \tag{3.4.5}$$

, where $k_{max} = max(k_1, k_2)$.

Similarly, smallest eigenvalue of a weight matrix can be written as

$$\lambda_{\frac{k_1}{2}, \frac{k_2}{2} \ldots \frac{k_m}{2}}(W) = (1 - 2mhr) + 2mh \sum_{j=1}^{r} \cos(\pi j), \tag{3.4.6}$$

Convergence parameter γ is minimum when

$$SLEM = \left|\lambda_{\frac{k_1}{2}, \frac{k_2}{2}, \ldots\ldots \frac{k_m}{2}}(W)\right|. \tag{3.4.7}$$

Substitution of (3.4.5) and (3.4.6) in (3.4.7) results in

$$h = \frac{1}{r(m+1) - \sum_{j=1}^{r} \cos\left(\frac{2\pi i}{k_{max}}\right) - m \sum_{j=1}^{r} \cos(\pi i)}, \tag{3.4.8}$$

where $k_{max} = max(k_1, k_2 \ldots k_m)$.

Using (3.2.18), h can be further simplified as

$$h = \frac{1}{(m+1)(r+0.5) - \frac{0.5 \sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} - \frac{m \cos \pi r}{2}}. \tag{3.4.9}$$

Substitution of (3.4.9) in (3.4.5) results in γ as

$$\gamma = \frac{(m-1)(r+0.5) + \frac{0.5 \sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} - \frac{m}{2}\cos(\pi r)}{(m+1)(r+0.5) - \frac{0.5 \sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} - \frac{m}{2}\cos(\pi r)}. \tag{3.4.10}$$

Finally, substitution of γ in (2.3.1) proves the theorem.

■

Theorem 3.4.3. *Convergence rate of a m-dimensional r-nearest neighbor torus for* $k_1 = k_2 = ... = k_m = odd$ *is expressed as*

$$R = 1 - \left| \frac{(m-1)(r+0.5) + \frac{1}{2}\left(\frac{\sin\frac{(r+0.5)2\pi}{k_{max}}}{\sin\frac{\pi}{k_{max}}} - \sum_{l=1}^{m} \frac{\sin\frac{(r+0.5)\pi(k_l-1)}{k_l}}{\sin\frac{\pi(k_l-1)}{2k_l}} \right)}{(m+1)(r+0.5) - \frac{1}{2}\left(\frac{\sin\frac{(r+0.5)2\pi}{k_{max}}}{\sin\frac{\pi}{k_{max}}} + \sum_{l=1}^{m} \frac{\sin\frac{(r+0.5)\pi(k_l-1)}{k_l}}{\sin\frac{\pi(k_l-1)}{2k_l}} \right)} \right|. \tag{3.4.11}$$

Proof. From (3.4.1), second largest eigenvalue of a weight matrix can be written as

$$(1 - 2hr) + 2h \sum_{j=1}^{r} \cos\left(\frac{2\pi j}{k_1}\right), \tag{3.4.12}$$

Similarly, smallest eigenvalue of a weight matrix can be written as

$$\lambda_{\frac{(k_1-1)}{2}, \frac{(k_2-1)}{2} .... \frac{(k_m-1)}{2}}(W) = (1 - 2mhr) + 2h \sum_{l=1}^{m} \sum_{j=1}^{r} \cos\left(\frac{\pi j(k_l-1)}{k_l}\right). \tag{3.4.13}$$

Convergence parameter γ is minimum when

$$SLEM = \left| \lambda_{\frac{(k_1-1)}{2}, \frac{(k_2-1)}{2}, ..., \frac{(k_m-1)}{2}}(W) \right|. \tag{3.4.14}$$

Finally, substitution of (3.4.12) and (3.4.13) in (3.4.14) results in

$$h = \frac{1}{r(m+1) - \sum_{l=1}^{r} \cos\left(\frac{2\pi l}{k_{max}}\right) - \sum_{j=1}^{m} \sum_{i=1}^{r} \cos\left(\frac{\pi i(k_j-1)}{k_j}\right)}. \tag{3.4.15}$$

Using (3.2.18), h can be further simplified as

$$h = \frac{1}{(m+1)(r+0.5) - \frac{0.5\sin\left(\frac{(2r+1)\pi}{k_{max}}\right)}{\sin\left(\frac{\pi}{k_{max}}\right)} - \sum_{l=1}^{m} \frac{0.5\sin\left(\frac{(2r+1)\pi(k_l-1)}{2k_l}\right)}{\sin\left(\frac{\pi(k_l-1)}{2k_l}\right)}}. \tag{3.4.16}$$

Substitution of (3.4.16) in (3.4.12) results in γ as

$$\gamma = \frac{(m-1)(r+0.5) + \frac{1}{2}\left(\frac{\sin\frac{(r+0.5)2\pi}{k_{max}}}{\sin\frac{\pi}{k_{max}}} - \sum_{l=1}^{m} \frac{\sin\frac{(r+0.5)\pi(k_l-1)}{k_l}}{\sin\frac{\pi(k_l-1)}{2k_l}} \right)}{(m+1)(r+0.5) - \frac{1}{2}\left(\frac{\sin\frac{(r+0.5)2\pi}{k_{max}}}{\sin\frac{\pi}{k_{max}}} + \sum_{l=1}^{m} \frac{\sin\frac{(r+0.5)\pi(k_l-1)}{k_l}}{\sin\frac{\pi(k_l-1)}{2k_l}} \right)}. \tag{3.4.17}$$

Finally, substitution of (3.4.17) in (2.3.1) completes the proof. ■

3.5 Numerical Results and Discussion

In this section, we present the numerical results to examine the effect of n, m, and r on convergence rate of average consensus algorithms. We have varied the network size from n= 22 to 500 and r from 2 to 10. Plots of the h versus n for the r-nearest neighbor ring network are shown in Fig. 3.5. We can observe that h increases with n for small-scale networks. From $n = 40$, convergence rate becomes constant for large-scale networks. As r values are varied from 2 to 10, h decreases with the values of r. Increase in r results improves the connectivity in the network which helps to reach the convergence rate with low h values. Plots of γ versus n are shown in the Fig. 3.6. We can notice that for small scale networks, γ is increasing with n in small-scale networks. As shown in the Fig. 3.6, γ values approaches to unity for large-scale networks. However, it has been observed that increase in node's transmission radius decreases the γ values. To examine the effect of n and r on convergence rate, Fig. 3.7 has been plotted. We can observe that, convergence rate of average consensus algorithms decreases with the number of nodes exponentially and increases with the node's transmission radius. Hence, it is essential to increase the node's transmission radius for fast convergence rates in large-scale WSNs. Fig. 3.8, Fig .3.9, and Fig. 3.10 have been plotted to investigate the effect of number of nodes on h, γ, and R respectively in a torus network. It has been observed that, consensus parameter increases with the n initially and becomes constant for high values of n. Convergence parameter becomes constant and approaches unity for large-scale WSNs. As shown in the Fig. 3.10, convergence rate decreases with the increase in the values of k_1 and k_2. Fig. 3.11, Fig. 3.12, and Fig. 3.13 have been plotted to understand the effect of network dimension on consensus parameter, convergence parameter, and convergence rate respectively. We observe that, consensus parameter decreases with the network dimension exponentially. Convergence parameter increases with the network dimension and approaches unity for large-scale WSNs. From Fig. 3.13, we can see that convergence rate decreases with the network dimension exponentially. However, for high r values convergence rates are significantly improved.

Convergence rate and Transmission Radius Optimization

Numerical results demonstrated that nearest neighbors or node's transmission radius increases the convergence rate, which is a primary objective of consensus algorithm. However, the node's power consumption [48] is

$$P = \left(\frac{r}{\sqrt{n}}\right)^{\alpha}$$

where α is a path-loss exponent. Node's transmission radius is directly proportional to node's power consumption. Increase in r values improve the convergence rate of average consensus algorithm significantly but at the cost of node's power consumption. Hence, it is essential to compute the optimal transmission radius in large-scale WSNs which can improve the convergence rate by considering the node's power consumption.

$$\begin{aligned} &\text{maximize} \quad R \\ &\text{subject to} \quad P \leq P_{max}, \quad n \leq n_{max}, \end{aligned}$$

where P_{max} is a maximum power consumption defined based on the WSN resource requirements.

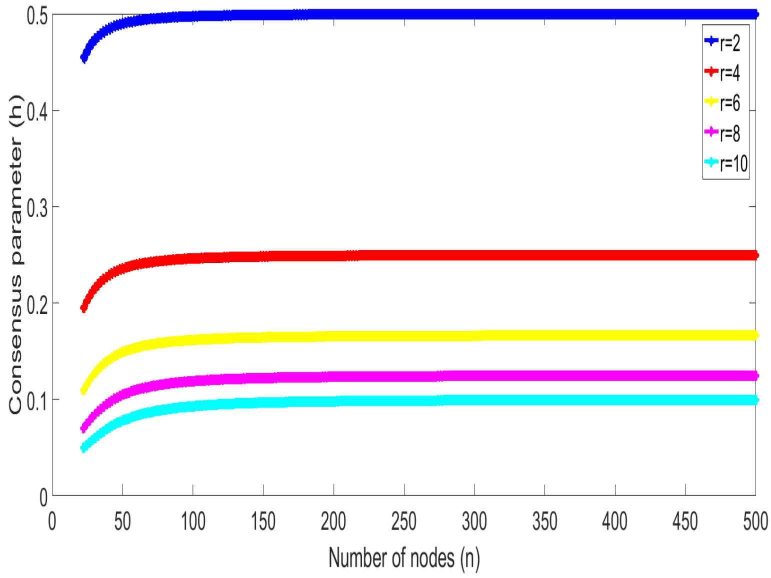

Figure 3.5: Effect of number of nodes on consensus parameter for r-nearest neighbor ring network.

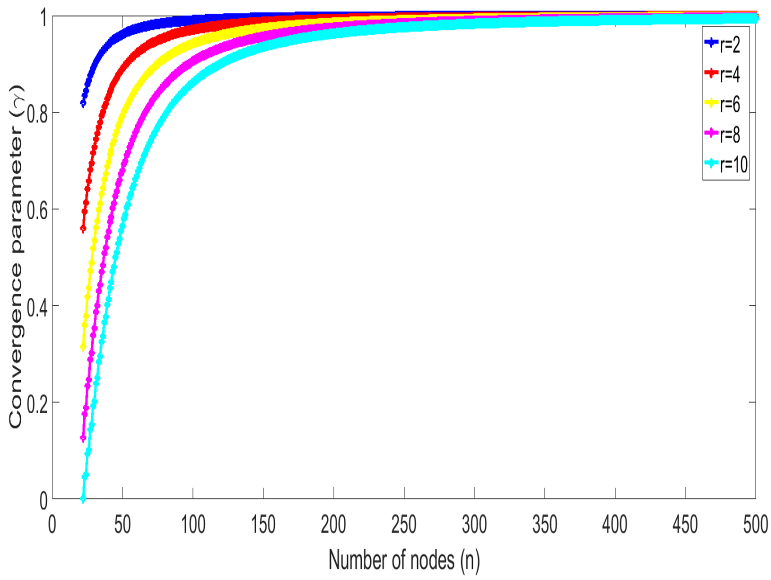

Figure 3.6: Effect of number of nodes on convergence parameter for r-nearest neighbor ring network.

3.6 Conclusions

In this chapter, the analytic expressions for optimal consensus parameter, optimal convergence parameter have been derived to estimate the convergence rate of m-

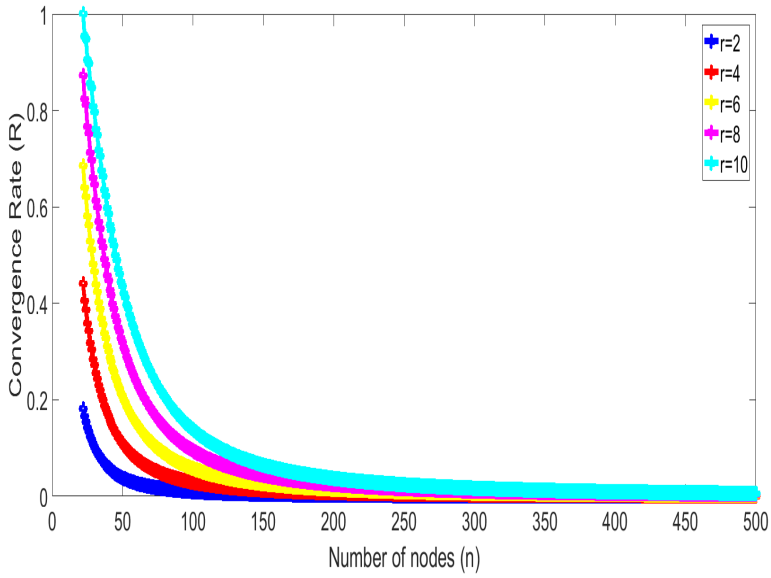

Figure 3.7: Effect of number of nodes on convergence rate for r-nearest neighbor ring network.

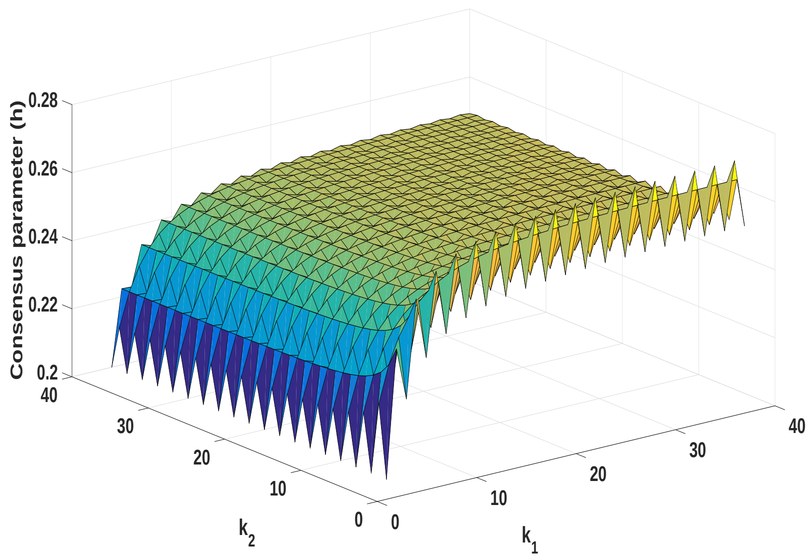

Figure 3.8: Effect of k_1 and k_2 on consensus parameter for torus network for r=1.

dimensional WSNs. It has been investigated that nodes in multidimensional WSNs require more nearest neighbors or large transmission radius without affecting the power consumption. An optimization framework has been proposed to design and control the

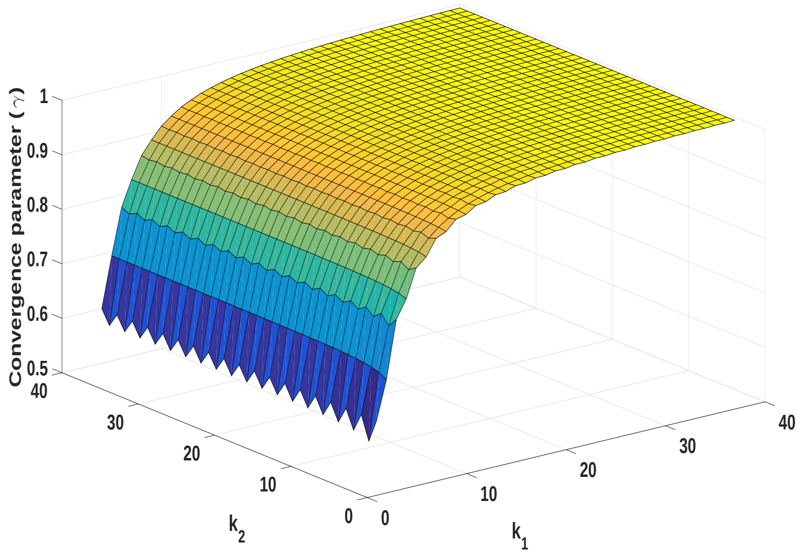

Figure 3.9: Effect of k_1 and k_2 on convergence parameter for torus network for r=1.

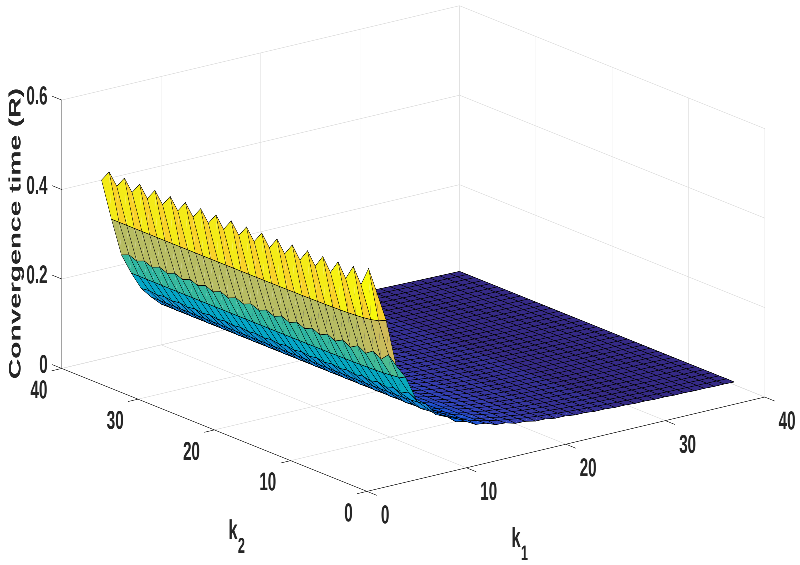

Figure 3.10: Effect of k_1 and k_2 on convergence rate for torus network for r=1.

performance of the consensus algorithm on WSNs. Furthermore, the analytic expressions derived in this chapter are extremely useful to exactly estimate the convergence rate for large WSNs with less computational complexity.

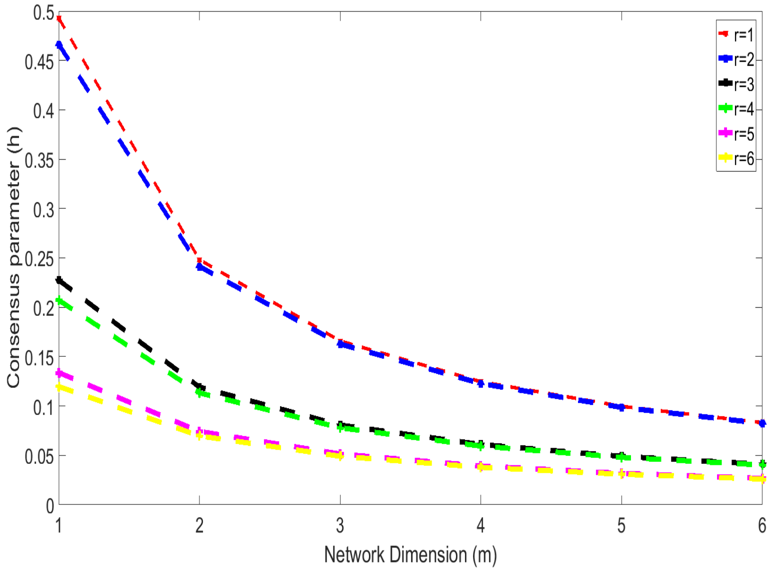

Figure 3.11: Effect of network dimension on consensus parameter for m-dimensional torus network for $k_1 = 16$, $k_2 = 18$, $k_3 = 20$, $k_4 = 22$, $k_5 = 24$ and $k_6 = 26$.

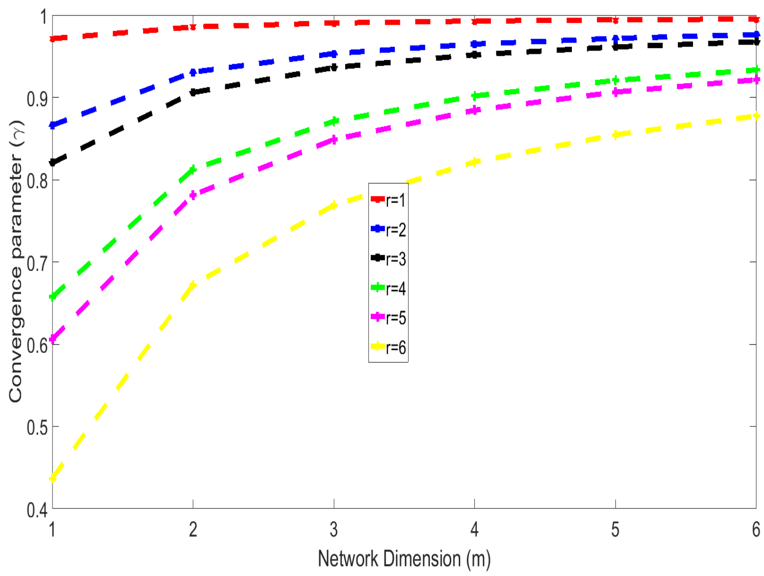

Figure 3.12: Effect of network dimension on convergence parameter for m-dimensional torus network for $k_1 = 16$, $k_2 = 18$, $k_3 = 20$, $k_4 = 22$, $k_5 = 24$ and $k_6 = 26$.

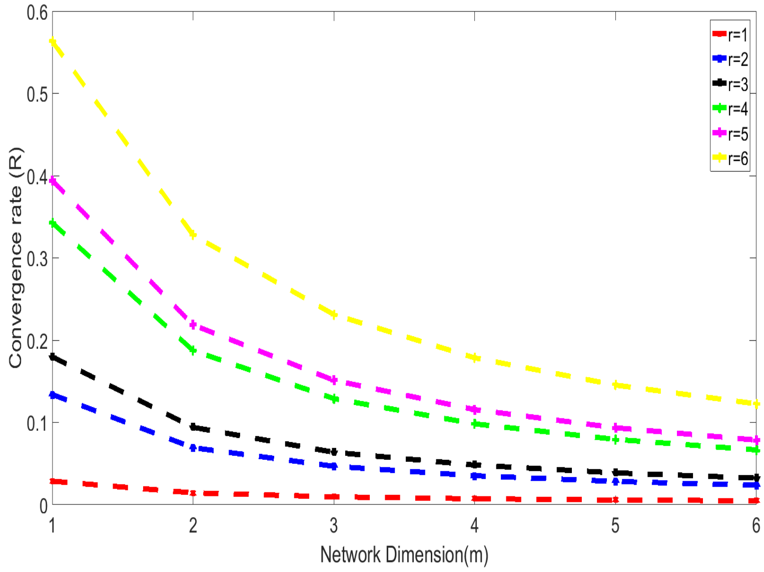

Figure 3.13: Effect of network dimension on convergence rate for m-dimensional torus network for $k_1 = 16$, $k_2 = 18$, $k_3 = 20$, $k_4 = 22$, $k_5 = 24$ and $k_6 = 26$.

Chapter 4

Analysis of Average Consensus Algorithms for Asymmetric Regular Networks

4.1 Introduction

Convergence rate of the consensus algorithms has been widely studied in the literature. However, most of the prior works have modeled the networks as an undirected graph due to the computational tractability [7], [48], [30], [46], [49], [50]. The undirected graph cannot model the applications which involve asymmetric links and may not characterize the actual networks performance. In practice, wireless channels in low power wireless networks such as WSNs are known to be time-varying, unreliable, and asymmetric [51], [52], [53], [54], [55], [56], [57]. Therefore, it is important to consider the WSN as a directed graph to accurately estimate the convergence rate. Convergence rate is characterized by the second largest eigenvalue of Laplacian matrix [58], [59]. But, determining the convergence rate for large-scale networks is a computationally challenging task. To evaluate the convergence rate, there are many algorithms available in the literature, such as *best constant* weights algorithm, *metropolis-hastings* weights algorithm, *max-degree* weights algorithm [7].

In this chapter, we use the *best constant* weights algorithm to derive the explicit expressions of convergence rate. In [49], authors modeled the WSN as an r-nearest neighbor network and derived the explicit expressions for convergence time of average consensus algorithms. However, they considered the undirected graph model which cannot study the time-varying wireless channels of WSNs. In [60], authors modeled the WSN as a directed graph and derived the explicit expressions of convergence rate for m-dimensional lattice networks. In [58], authors modeled the network as a weighted directed graph and studied the expected rate of convergence in an asymmetric network.

In this chapter, we model the WSN as a directed graph and derive the explicit expressions of convergence rate [2] for regular graphs. Regular graph models are simple structures which allow the theoretical analysis that incorporates important parameters like connectivity, scalability, network size, node overhead, and network dimension [48], [61], [49]. These models represent the geographical proximity in the practical wireless sensor networks. In this chapter, we model the WSN as a ring, torus, r-nearest neighbor network, m-dimensional torus networks and derive the explicit expressions for convergence rate. Node's transmission radius or node overhead can be modeled by the nearest neighbors. Finally, we measure the absolute error to examine the accuracy of directed graph modeling over undirected graph modeling.

Organization

This chapter is organized as follows. In section 4.2, we model the WSN as a ring and derive the explicit expressions of convergence rate in terms of the number of nodes

[2] **Sateeshkrishna Dhuli** and Yatindra. Nath. Singh, "Analysis of Average Consensus Algorithm for Asymmetric Regular Networks," *arXiv preprint arXiv:1806.03932* , 2018.

and network overhead. In section 4.3, we model the WSN as a torus network and m-dimensional torus network and derive the explicit expressions of convergence rate in terms of the number of nodes and network dimension. In section 4.4, we model the WSN as a r-neighbor ring network and derive the explicit expressions of convergence rate in terms of nearest neighbors and number of nodes. In section 4.5, we present the numerical results and study the effect of network parameters on the convergence rate. Finally, we present the conclusions of this chapter in section 4.6.

4.2 Explicit Formulas of Convergence rate for Ring Networks

In this section, we derive the explicit expressions of convergence rate for a ring network. Ring network with asymmetric links is as shown in the Fig. 4.1. We assume that forward link weight is $\frac{1-a}{2}$ and backward link weight is $\frac{1+a}{2}$, here 'a' denotes the asymmetric link factor. In this chapter, we model the WSN as a weighted graph. Let $G = (V, E)$, be a directed graph with node set $V = \{1, 2, ...n\}$ and an edge set $E \subseteq V \times V$. In a weighted network, the entries [62] of the Laplacian matrix G is defined as

$$L_{i,j} = \begin{cases} -l_{i,j} & i \neq j \quad (i,j) \in E, \\ \sum_{k=1}^{N} l_{i,k} & i = j \quad (i,k) \in E, \\ 0 & Otherwise \end{cases} \tag{4.2.1}$$

where $l_{i,k}$ represents the link weight between node i and k.

Theorem 4.2.1. *The $(j+1)^{th}$ eigenvalue of Laplacian matrix for a ring network is expressed as*

$$\lambda_j(L) = 1 - \cos\frac{2\pi j}{n} + ai\sin\frac{2\pi j}{n} \tag{4.2.2}$$

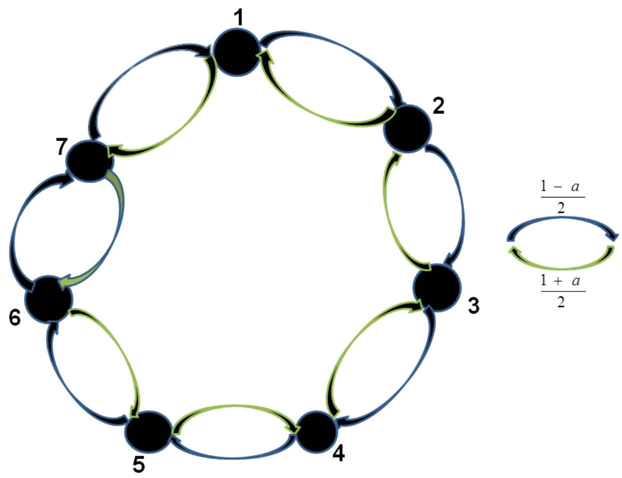

Figure 4.1: Asymmetric Ring Network

Proof. Asymmetric ring network is as shown in the Fig. 4.1. Thus, Laplacian matrix can be written as

$$L = circ(1, \frac{-1+a}{2}, \underbrace{0, 0, .., 0}_{n-3\ terms}, \frac{-1-a}{2}) \tag{4.2.3}$$

Applying (3.2.2), $(j+1)^{th}$ eigenvalue of the Laplacian matrix can be written as

$$\lambda_j(L) = 1 - \cos\frac{2\pi j}{n} + ai\sin\frac{2\pi j}{n} \tag{4.2.4}$$

■

Theorem 4.2.2. *Convergence rate of a ring network for even number of nodes is expressed as*

$$R = 1 - \left|\frac{2 - 2a^2 - 2\cos\frac{2\pi}{n} + 2a^2\cos\frac{2\pi}{n}}{3 - a^2 + (-1+a^2)\cos\frac{2\pi}{n}}\right|. \tag{4.2.5}$$

Proof. For n=even, $\lambda_1(L)$ is the second smallest value of Laplacian matrix and $\lambda_{\frac{n}{2}}(L)$ is the largest of the Laplacian matrix.

From best-constant algorithm, γ is minimum, when

$$|1 - h\lambda_1(L)| = \left|1 - h\lambda_{\frac{n}{2}}(L)\right| \tag{4.2.6}$$

Substituting the expressions of $\lambda_1(L)$ and $\lambda_{\frac{n}{2}}(L)$ in (4.2.6), gives the

$$h = \frac{2 + 2\cos\frac{2\pi}{n}}{3 - \cos^2\frac{2\pi}{n} + 2\cos\frac{2\pi}{n} - a^2\sin^2\frac{2\pi}{n}}. \tag{4.2.7}$$

Thus, convergence factor (γ) is expressed as

$$\gamma = \left|1 - h\left(1 - \cos\left(\frac{2\pi}{n}\right) + ia\sin\left(\frac{2\pi}{n}\right)\right)\right| \tag{4.2.8}$$

Substituting the γ in (2.3.1) completes the proof. ■

Theorem 4.2.3. *Convergence rate of a ring network for odd number of nodes is expressed as*

$$R = 1 - \left|\frac{\sqrt{2+4a^2+2a^4-2(-1+a^4)\cos\frac{\pi}{n}+(-1+a^2)^2\cos\frac{2\pi}{n}+2\cos\frac{3\pi}{n}-2a^4\cos\frac{3\pi}{n}+\cos\frac{4\pi}{n}-2a^2\cos\frac{4\pi}{n}+a^4\cos\frac{4\pi}{n}}}{\sqrt{2}\left(2-(-1+a^2)\cos\frac{\pi}{n}+(-1+a^2)\cos\frac{2\pi}{n}\right)}\right|. \tag{4.2.9}$$

Proof. For n=odd, $\lambda_1(L)$ is the second smallest eigenvalue of Laplacian matrix and $\lambda_{\frac{n-1}{2}}(L)$ is the largest eigenvalue of a Laplacian matrix.

Thus, from best constant algorithm γ is minimum when

$$|1 - h\lambda_1(L)| = \left|1 - h\lambda_{\frac{n-1}{2}}(L)\right| \tag{4.2.10}$$

Substituting the $\lambda_1(L)$ and $\lambda_{\frac{n-1}{2}}(L)$ expressions in (4.2.10) results

$$\left|1 - h\left(1 - \cos\tfrac{2\pi}{n} + ia\sin\tfrac{2\pi}{n}\right)\right| = \left|1 - h\left(1 + \cos\tfrac{\pi}{n} + ia\sin\tfrac{\pi}{n}\right)\right| \tag{4.2.11}$$

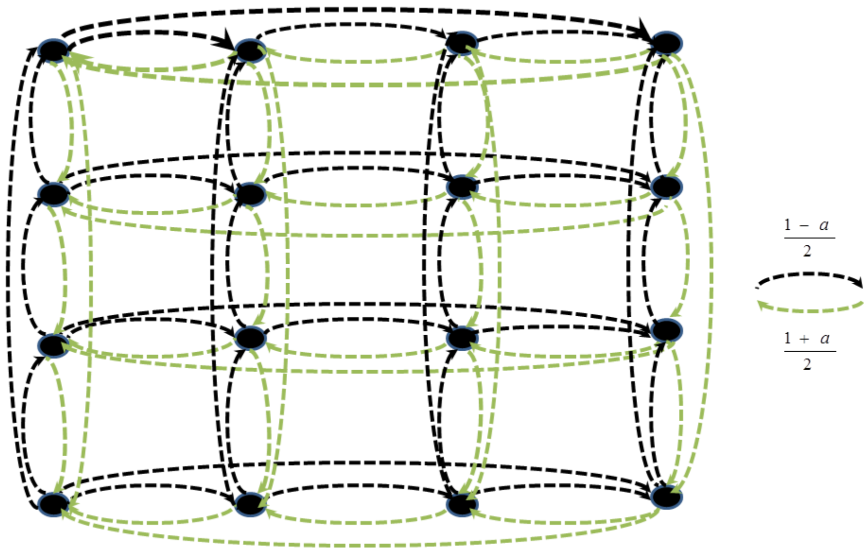

Figure 4.2: Asymmetric Torus Network

Thus, we obtain

$$h = \frac{2\left(\cos\frac{\pi}{n}+\cos\frac{2\pi}{n}\right)}{-\cos^2\frac{2\pi}{n}+2\cos\frac{2\pi}{n}-a^2\sin^2\frac{2\pi}{n}+\cos^2\frac{\pi}{n}+a^2\sin^2\frac{\pi}{n}+2\cos\frac{\pi}{n}} \tag{4.2.12}$$

Finally, we get γ as

$$\gamma = \left|1-h\left(1-\cos\left(\frac{2\pi}{n}\right)+ia\sin\left(\frac{2\pi}{n}\right)\right)\right| \tag{4.2.13}$$

Substituting the (4.2.13) value in (2.3.1) completes the proof. ■

4.3 Explicit Formulas of Convergence rate for Torus Networks

Torus network with asymmetric links is shown in Fig. 4.2. In this section, we derive the explicit expressions of convergence rate for a torus network and m-dimensional torus networks.

Theorem 4.3.1. *The eigenvalue of a torus network is expressed as*

$$\lambda_{j_1,j_2}(L) = 2 - \cos\tfrac{2\pi j_1}{k_1} - \cos\tfrac{2\pi j_2}{k_2} + ia\left(\sin\tfrac{2\pi j_1}{k_1} + \sin\tfrac{2\pi j_2}{k_2}\right). \tag{4.3.1}$$

Proof. A torus network is formed by the cartesian product of two ring networks. The eigenvalue of a torus network will be the addition of eigenvalues of the corresponding ring networks [28].

$$\lambda_{j_1,j_2}(L) = \lambda_{j_1}(L) + \lambda_{j_2}(L) \tag{4.3.2}$$

Here, we assume that the torus is formed by two ring networks with k_1 and k_2 nodes respectively. Then $(j_1+1)^{th}$ eigenvalue of the Laplacian matrix for a ring network can be expressed as

$$\lambda_{j_1}(L) = 1 - \cos\frac{2\pi j_1}{k_1} + ia\sin\frac{2\pi j_1}{k_1} \tag{4.3.3}$$

Similarly, $(j_2+1)^{th}$ eigenvalue of the Laplacian matrix for ring network is expressed as

$$\lambda_{j_2}(L) = 1 - \cos\frac{2\pi j_2}{k_2} + ia\sin\frac{2\pi j_2}{k_2} \tag{4.3.4}$$

Finally, we obtain (4.3.1) using (4.3.2), (4.3.3), and (4.3.4) . ■

Theorem 4.3.2. *Convergence rate of a torus network for $k_1 = even$ and $k_2 = even$ is expressed as*

$$R = 1 - \left| \frac{a^2 \sin^2\left(\frac{2\pi}{k_{max}}\right) + \cos^2\left(\frac{2\pi}{k_{max}}\right) + 6\cos\left(\frac{2\pi}{k_{max}}\right) + 9}{a^2 \sin^2\left(\frac{2\pi}{k_{max}}\right) + \cos^2\left(\frac{2\pi}{k_{max}}\right) - 2\cos\left(\frac{2\pi}{k_{max}}\right) - 15} \right|, \tag{4.3.5}$$

where k_{max}=$max(k_1, k_2)$.

Proof. For $k_1 = even$ and $k_2 = even$, $\lambda_{\frac{k_1}{2},\frac{k_2}{2}}(L)$ is the largest eigenvalue of the Laplacian matrix. Thus, γ is minimum when

$$SLEM = \left|1 - h\lambda_{\frac{k_1}{2},\frac{k_2}{2}}(L)\right| \tag{4.3.6}$$

Substituting the second smallest eigenvalue of Laplacian matrix and $\lambda_{\frac{k_1}{2},\frac{k_2}{2}}(L)$ in (4.3.6), results in

$$\left|1 - h\left(1 - \cos\frac{2\pi}{k_{max}} + ia\sin\frac{2\pi}{k_{max}}\right)\right| = |1 - 4h| \tag{4.3.7}$$

where k_{max}=$max(k_1, k_2)$.

Simplifying (4.3.7) to obtain

$$h = \frac{6 + 2\cos\frac{2\pi}{k_{max}}}{15 - \cos^2\frac{2\pi}{k_{max}} + 2\cos\frac{2\pi}{k_{max}} - a^2\sin^2\frac{2\pi}{k_{max}}} \tag{4.3.8}$$

Finally, we obtain the convergence parameter γ as

$$\gamma = \left|1 - h\left(1 - \cos\frac{2\pi}{k_{max}} + ia\sin\frac{2\pi}{k_{max}}\right)\right| \tag{4.3.9}$$

Thus, substituting (4.3.9) in (2.3.1) proves the *Theorem*.

■

Theorem 4.3.3. *Convergence rate of a torus network for $k_1 = odd$ and $k_2 = odd$ is expressed as*

$$R = 1 - \left| \sqrt{\frac{a^2 p_1^2 \sin^2 \frac{2\pi}{k_{max}}}{q_1^2} + \left(1 - \frac{p_1 \sin^2 \frac{\pi}{k_{max}}}{q_1}\right)^2} \right|. \tag{4.3.10}$$

where

$$p_1 = 4\left(2\cos\left(\frac{\pi}{k_1}\right) + \cos\left(\frac{\pi}{k_2}\right) + \cos\left(\frac{2\pi}{k_2}\right) + 1\right),$$

$$\begin{aligned} q_1 = & -a^2 \sin^2\left(\frac{2\pi}{k_2}\right) + a^2\left(\sin\left(\frac{\pi}{k_1}\right) + \sin\left(\frac{\pi}{k_2}\right)\right)^2 \\ & - \cos^2\left(\frac{2\pi}{k_2}\right) + \left(2\cos\left(\frac{\pi}{k_1}\right) + \cos\left(\frac{\pi}{k_2}\right)\right)^2 \\ & + 8\cos\left(\frac{\pi}{k_1}\right) + 4\cos\left(\frac{\pi}{k_2}\right) + 2\cos\left(\frac{2\pi}{k_2}\right) + 3, \end{aligned}$$

where k_{max}=$max(k_1, k_2, ...k_m)$.

Proof. For $k_1 = odd$ and $k_2 = odd$, $\lambda_{1,0}(L)$ is the second smallest eigenvalue and $\lambda_{\frac{k_1-1}{2},\frac{k_2-1}{2}}(L)$ is the largest eigenvalue of the Laplacian matrix. Thus, γ is minimum when

$$SLEM = \left|1 - h\lambda_{\frac{k_1-1}{2},\frac{k_2-1}{2}}(L)\right| \tag{4.3.11}$$

Substitute the expressions of second largest eigenvalue of Laplacian matrix and $\lambda_{\frac{k_1-1}{2},\frac{k_2-1}{2}}(L)$ in (4.3.11) results in

$$\left|1 - h\left(1 - \cos\tfrac{2\pi}{k_{max}} + ia\sin\tfrac{2\pi}{k_{max}}\right)\right| = \left|1 - h\left(2 - \cos\tfrac{\pi(k_1-1)}{k_1} - \cos\tfrac{\pi(k_2-1)}{k_2} + ia\left(\sin\tfrac{\pi(k_1-1)}{k_1} + \sin\tfrac{\pi(k_2-1)}{k_2}\right)\right)\right| \tag{4.3.12}$$

where k_{max}=$max(k_1, k_2, ...k_m)$.

Thus, we get

$$h = \frac{-2\cos\left(\frac{2\pi}{k_{max}}\right)+2\left(2\cos\left(\frac{\pi(k_1-1)}{k_1}\right)+\cos\left(\frac{\pi(k_2-1)}{k_2}\right)\right)-2}{0.16\sin^2\left(\frac{2\pi}{k_{max}}\right)-0.16\left(\sin\left(\frac{\pi(k_1-1)}{k_1}\right)+\sin\left(\frac{\pi(k_{max}-1)}{k_{max}}\right)\right)^2+\cos^2\left(\frac{2\pi}{k_{max}}\right)-2\cos\left(\frac{2\pi}{k_{max}}\right)-\left(2\cos\left(\frac{\pi(k_1-1)}{k_1}\right)+\cos\left(\frac{\pi(k_2-1)}{k_2}\right)\right)^2+4\left(2\cos\left(\frac{\pi(k_1-1)}{k_1}\right)+\cos\left(\frac{\pi(k_2-1)}{k_2}\right)\right)-3} \tag{4.3.13}$$

Finally, we obtain the convergence parameter γ as

$$\gamma = \left|1 - h\left(1 - \cos\frac{2\pi}{k_{max}} + ia\sin\frac{2\pi}{k_{max}}\right)\right| \tag{4.3.14}$$

Substituting the (4.3.14) in (2.3.1) results in (4.3.10). ■

Theorem 4.3.4. *The eigenvalue of an m-dimensional torus network is expressed as*

$$\lambda_{j_1,j_2.....j_m}(L) = m - \sum_{l=1}^{m}\cos\frac{2\pi j_l}{k_l} + ia\left(\sum_{l=1}^{m}\sin\frac{2\pi j_l}{k_l}\right). \tag{4.3.15}$$

Proof. Cartesian product of 'm' ring networks results in m-dimenional torus network. The eigenvalue of a torus network will be the addition of eigenvalues of corresponding 'm' ring networks [28].

$$\lambda_{j_1,j_2.....j_m}(L) = \lambda_{j_1}(L) + \lambda_{j_2}(L) + ... + \lambda_{j_1}(L) + \lambda_{j_m}(L) \tag{4.3.16}$$

Here, we assume that the torus is formed by cartesian product of 'm' ring networks with k_m nodes, $m = 1, 2.....$ Then $(j_1+1)^{th}$ eigenvalue of the Laplacian matrix for ring network with k_1 nodes can be expressed as

$$\lambda_{j_1}(L) = 1 - \cos\frac{2\pi j_1}{k_1} + ia\sin\frac{2\pi j_1}{k_1} \tag{4.3.17}$$

The $(j_2+1)^{th}$ eigenvalue of the Laplacian matrix for ring network with k_2 nodes is expressed as

$$\lambda_{j_2}(L) = 1 - \cos\frac{2\pi j_2}{k_2} + ia\sin\frac{2\pi j_2}{k_2} \tag{4.3.18}$$

Similarly, $(j_m + 1)^{th}$ eigenvalue of a Laplacian matrix for ring network with k_m nodes is expressed as

$$\lambda_{j_m}(L) = 1 - \cos\frac{2\pi j_m}{k_m} + ia\sin\frac{2\pi j_m}{k_m} \tag{4.3.19}$$

We can write the eigenvalue of a m-dimensional torus network as (4.3.15) using (4.3.16), (4.3.17), (4.3.18), and (4.3.19) . ■

Theorem 4.3.5. *Convergence rate of an m-dimensional torus network for $k_1 = k_2 = ...k_m = even$ is expressed as*

$$R = 1 - \left|\frac{a^2\sin^2\left(\frac{2\pi}{k_1}\right)+(4m-2)\cos\left(\frac{2\pi}{k_1}\right)+\cos^2\left(\frac{2\pi}{k_1}\right)+(1-2m)^2}{a^2\sin^2\left(\frac{2\pi}{k_1}\right)+\cos^2\left(\frac{2\pi}{k_1}\right)-2\cos\left(\frac{2\pi}{k_1}\right)-4m^2+1}\right|. \tag{4.3.20}$$

Proof. For $k_1 = k_2 = ... = k_m = even$, largest eigenvalue of Laplacian matrix is $\lambda_{\frac{k_1}{2},\frac{k_2}{2},.....\frac{k_n}{2}}(L)$. Thus, γ is minimum when

$$SLEM = \left|1 - h\lambda_{\frac{k_1}{2},\frac{k_2}{2},.....\frac{k_n}{2}}(L)\right| \tag{4.3.21}$$

Substituting the second largest eigenvalue of Laplacian matrix and $\lambda_{\frac{k_1}{2},\frac{k_2}{2},.....\frac{k_n}{2}}(L)$ in (4.3.21) results in

$$SLEM = |1 - 2mh| \tag{4.3.22}$$

Thus, we obtain

$$h = \frac{2 - 2\cos\frac{2\pi}{k_{max}} - 4m}{1 - 4m^2 + \cos^2\frac{2\pi}{k_{max}} - 2\cos\frac{2\pi}{k_{max}} + a^2\sin^2\frac{2\pi}{k_{max}}} \tag{4.3.23}$$

Substituting the h in $\left|1 - h\left(1 - \cos\frac{2\pi}{k_{max}} + ia\sin\frac{2\pi}{k_{max}}\right)\right|$ gives γ. Finally, substituting γ value in (2.3.1) results in (4.3.20). ■

4.4 Explicit Formulas of Convergence rate for r-Nearest Neighbor Networks

In this section, we derive the explicit expressions of convergence rate for r-nearest neighbor networks. In this network, nodes with in the distance 'r' hops in the ring network are connected. The variable 'r' models the nodes' transmission radius or node overhead in WSNs.

Theorem 4.4.1. *The $(j+1)^{th}$ eigenvalue of a r-nearest neighbor ring network is expressed as*

$$\lambda_j(L) = r - \sum_{k=1}^{r} \cos\frac{2\pi jk}{n} + ia \sum_{k=1}^{r} \sin\frac{2\pi jk}{n} \tag{4.4.1}$$

Proof. Laplacian matrix of a r-nearest neighbor ring network with n can be written as

$$L = circ\left(r, \underbrace{\frac{-1+a}{2}, \frac{-1+a}{2}, ..., \frac{-1+a}{2}}_{r\ terms}, \underbrace{0, 0, ..., 0,}_{n-2r-1\ terms} \underbrace{\frac{-1-a}{2}, \frac{-1-a}{2}, ..., \frac{-1-a}{2}}_{r\ terms} \right) \tag{4.4.2}$$

From (3.2.2) and (4.4.2), we obtain (4.4.1). ■

Theorem 4.4.2. *Convergence rate of a r-nearest neighbor ring network for $n = even$ is expressed as*

$$R = 1 - \left| \sqrt{\left(\frac{p_2 q_2}{s_2}\right)^2 + \frac{r_2 q_2}{s_2} + 1} \right|. \tag{4.4.3}$$

, where

$$p_2 = \frac{a \sin\frac{\pi}{n} \cos\frac{(2r+1)\pi}{n} - 0.5a \sin\frac{2\pi}{n}}{\cos\frac{2\pi}{n} - 1},$$

$$q_2 = \frac{\sin\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}} - \cos\pi r,$$

$$r_2 = \frac{\sin\frac{\pi}{n}\sin\frac{(2r+1)\pi}{n}}{\cos\frac{2\pi}{n} - 1} + r + 0.5,$$

and

$$\begin{aligned} s_2 = &\frac{0.25a^2\left(2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{2\pi r+\pi}{n}\right) - \sin\left(\frac{2\pi}{n}\right)\right)^2}{\left(\cos\left(\frac{2\pi}{n}\right) - 1\right)^2} \\ &+ \frac{\sin^2\left(\frac{\pi}{n}\right)\sin^2\left(\frac{2\pi r+\pi}{n}\right)}{\left(\cos\left(\frac{2\pi}{n}\right) - 1\right)^2} \\ &+ (r+0.5)\left(\cos(\pi r) - \csc\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi r+\pi}{n}\right)\right) \\ &- 0.25\cos^2(\pi r). \end{aligned}$$

Proof. For n=even, $\lambda_1(L)$ is the second smallest value of Laplacian matrix and $\lambda_{\frac{n}{2}}(L)$ is the largest eigenvalue of the Laplacian matrix. Thus, γ is minimum when

$$|1 - h\lambda_1(L)| = \left|1 - h\lambda_{\frac{n}{2}}(L)\right| \tag{4.4.4}$$

Substituting the $\lambda_1(L)$ and $\lambda_{\frac{n}{2}}(L)$ in (4.4.4), results in

$$\left|1 - h\left(r + 0.5 - \frac{\sin\frac{(2r+1)\pi}{n}}{2\sin\frac{\pi}{n}} + \frac{ia}{2}\left(\cot\frac{\pi}{n} - \frac{\cos\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}}\right)\right)\right| = |1 - h\left(r + 0.5 - 0.5\cos\pi r\right)| \tag{4.4.5}$$

Thus, we get

$$h = \frac{\cos\pi r - \frac{\sin\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}}}{\frac{1}{4}\left(\frac{\sin\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}}\right)^2 - (r+0.5)\left(\frac{\sin\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}} - \cos\pi r\right) - \frac{\cos^2\pi r}{4} + \frac{e^2}{4}\left(\cot\frac{\pi}{n} - \frac{\cos\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}}\right)^2} \tag{4.4.6}$$

Thus, convergence factor is expressed as

$$\gamma = \left|1 - h\left(r + 0.5 - \frac{\sin\frac{(2r+1)\pi}{n}}{2\sin\frac{\pi}{n}} + \frac{ia}{2}\left(\cot\frac{\pi}{n} - \frac{\cos\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}}\right)\right)\right| \tag{4.4.7}$$

Substitute (4.4.7) in (2.3.1) proves the *Theorem*. ■

Theorem 4.4.3. *Convergence rate of a r-nearest neighbor ring network for $n = odd$ is expressed as*

$$R = 1 - \left|\sqrt{\left(\frac{p_3 q_3}{s_3}\right)^2 + \frac{r_3 q_3}{s_3} + 1}\right|. \tag{4.4.8}$$

, where

$$\begin{aligned}
p_3 &= \frac{-a\sin\frac{\pi}{n}\cos\frac{(2r+1)\pi}{n} + 0.5a\sin\frac{2\pi}{n}}{\cos\frac{2\pi}{n} - 1},\\
q_3 &= -\frac{\sin\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}} + \frac{\sin\frac{\pi(n-1)(2r+1)}{2n}}{\cos\frac{\pi}{2n}},\\
r_3 &= -\frac{\sin\frac{\pi}{n}\sin\frac{(2r+1)\pi}{n}}{\cos\frac{2\pi}{n} - 1} - r - 0.5,
\end{aligned}$$

and

$$\begin{aligned}
s_3 = &-\frac{0.25a^2\left(2\cos\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi(n-1)(2r+1)}{2n}\right) - \sin\left(\frac{\pi}{n}\right)\right)^2}{\left(\cos\left(\frac{\pi}{n}\right) + 1\right)^2}\\
&+ \frac{0.25a^2\left(2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{2\pi r+\pi}{n}\right) - \sin\left(\frac{2\pi}{n}\right)\right)^2}{\left(\cos\left(\frac{2\pi}{n}\right) - 1\right)^2}\\
&- \frac{\cos^2\left(\frac{\pi}{2n}\right)\sin^2\left(\frac{\pi(n-1)(2r+1)}{2n}\right)}{\left(\cos\left(\frac{\pi}{n}\right) + 1\right)^2} + \frac{\sin^2\left(\frac{\pi}{n}\right)\sin^2\left(\frac{2\pi r+\pi}{n}\right)}{\left(\cos\left(\frac{2\pi}{n}\right) - 1\right)^2}\\
&+ (r + 0.5)\left(\frac{\sin\frac{\pi(n-1)(2r+1)}{2n}}{\cos\frac{\pi}{2n}} - \frac{\sin\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}}\right).
\end{aligned}$$

Proof. For n=odd, $\lambda_1(L)$ is the second smallest eigenvalue and $\lambda_{\frac{n-1}{2}}(L)$ is the largest eigenvalue of the Laplacian matrix.

Thus, γ is minimum when

$$|1 - h\lambda_1(L)| = \left|1 - h\lambda_{\frac{n-1}{2}}(L)\right| \tag{4.4.9}$$

After substituting the $\lambda_1(L)$ and $\lambda_{\frac{n-1}{2}}(L)$ expressions in (4.4.9), we obtain

$$\left|1 - h\left(r + 0.5 - \frac{\sin\frac{(2r+1)\pi}{n}}{2\sin\frac{\pi}{n}} + \frac{ia}{2}\left(\cot\frac{\pi}{n} - \frac{\cos\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}}\right)\right)\right| = \left|1 - h\left(r + 0.5 - \frac{\sin\frac{(2r+1)\pi(n-1)}{2n}}{\sin\frac{\pi(n-1)}{2n}} + \frac{ia}{2}\left(\cot\frac{\pi(n-1)}{2n} - \frac{\cos\frac{(2r+1)\pi(n-1)}{2n}}{\sin\frac{\pi(n-1)}{2n}}\right)\right)\right| \tag{4.4.10}$$

Thus, we obtain

$$h = \frac{\frac{\sin\frac{(2r+1)\pi(n-1)}{2n}}{\sin\frac{\pi(n-1)}{2n}} - \frac{\sin\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}}}{\frac{1}{4}\left(\frac{\sin\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}}\right)^2 + (r+0.5)\left(-\frac{\sin\frac{(2r+1)\pi(n-1)}{2n}}{\sin\frac{\pi(n-1)}{2n}} + \frac{\sin\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}}\right) - \frac{1}{4}\left(\frac{\sin\frac{(2r+1)\pi(n-1)}{2n}}{\sin\frac{\pi(n-1)}{2n}}\right)^2 + \frac{e^2}{4}\left(\cot\frac{\pi}{n} - \frac{\cos\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}}\right)^2 - \frac{e^2}{4}\left(\cot\frac{\pi(n-1)}{2n} - \frac{\cos\frac{(2r+1)\pi(n-1)}{2n}}{\sin\frac{\pi(n-1)}{2n}}\right)^2} \tag{4.4.11}$$

Finally, convergence factor γ is expressed as

$$\gamma = \left|1 - h\left(r + 0.5 - \frac{\sin\frac{(2r+1)\pi}{n}}{2\sin\frac{\pi}{n}} + \frac{ia}{2}\left(\cot\frac{\pi}{n} - \frac{\cos\frac{(2r+1)\pi}{n}}{\sin\frac{\pi}{n}}\right)\right)\right| \tag{4.4.12}$$

Substituting the (4.4.12) in (2.3.1) proves the *Theorem*. ■

4.5 Numerical Results and Discussion

In this section, we present the numerical results to investigate the effect of asymmetric link factor, network dimension, number of nodes, and node overhead on the convergence rate of the average consensus algorithm. We have used the *Wolfram Mathematica* to solve the analytical expressions. Fig. 4.3 shows the comparison of convergence rates of average consensus algorithms for asymmetric and symmetric ring networks. We have observed that the convergence rate decreases exponentially with both the number of nodes and asymmetric link factor. In large-scale WSNs, convergence rate approaches

0 as shown in the Fig. 4.3. Fig. 4.4 shows the convergence rate versus k_1 and k_2 in torus network. Here, convergence rate decreases with k_1 and k_2 exponentially. Fig. 4.5 shows the convergence rate versus asymmetric link factor for different values of r. We have noted that the convergence rate increases with the nodes' transmission radius and decreases with asymmetric link factor. To understand the effect of network dimension on the convergence rate, we plotted the Fig. 4.6. We have observed that the convergence rate decreases with the network dimension. To compute the error introduced by the symmetric network modeling, we compute the absolute error $R_s - R_a$, where R_s and R_a denote the convergence rates of symmetric and asymmetric networks respectively. Fig. 4.7 shows the Absolute Error versus Number of nodes. Here, the absolute error decreases with the number of nodes. Also, the absolute error is significant for large values of asymmetric link factors. The effect of asymmetric link modeling on the convergence rate is high in small-scale networks.

4.6 Conclusions

In this chapter, we modeled the WSN as a directed graph and derived the explicit formulas for the ring, torus, r-nearest neighbor ring, and m-dimensional torus networks. Numerical results demonstrated that the convergence rate decreases significantly with the increase of asymmetrical link factor in small-scale WSNs. In large-scale WSNs, the effect of asymmetrical links on convergence rate exponentially decreases with the number of nodes. Further, we have studied the impact of the network size, network dimension, and nodes' transmission radius on the convergence rate of average consensus algorithms.

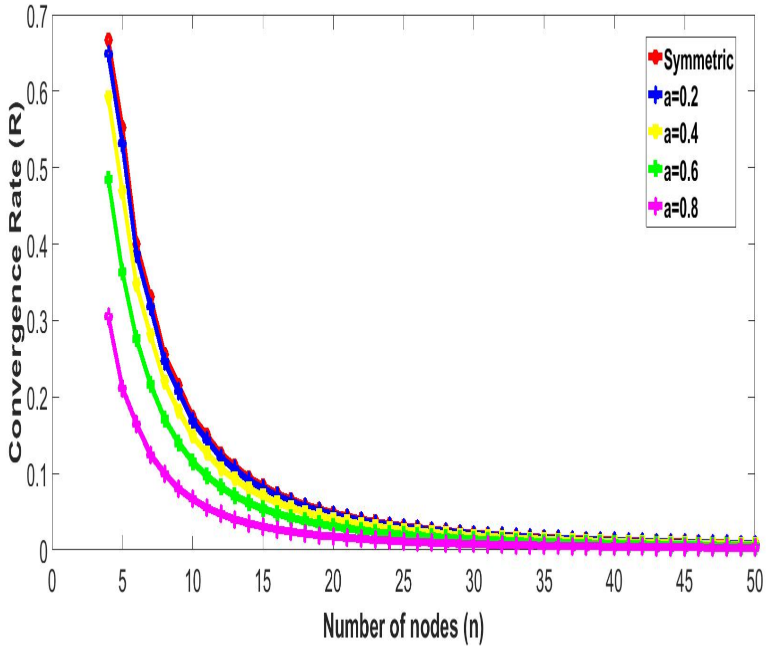

Figure 4.3: Comparison of convergence rates in asymmetric and symmetric ring networks.

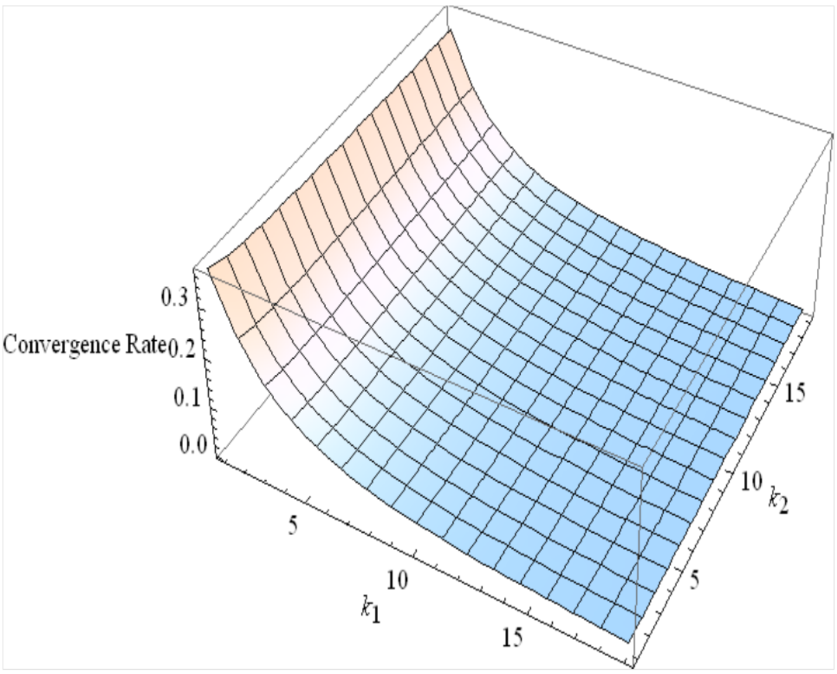

Figure 4.4: Convergence rate versus k_1 and k_2 for a torus network for n=odd.

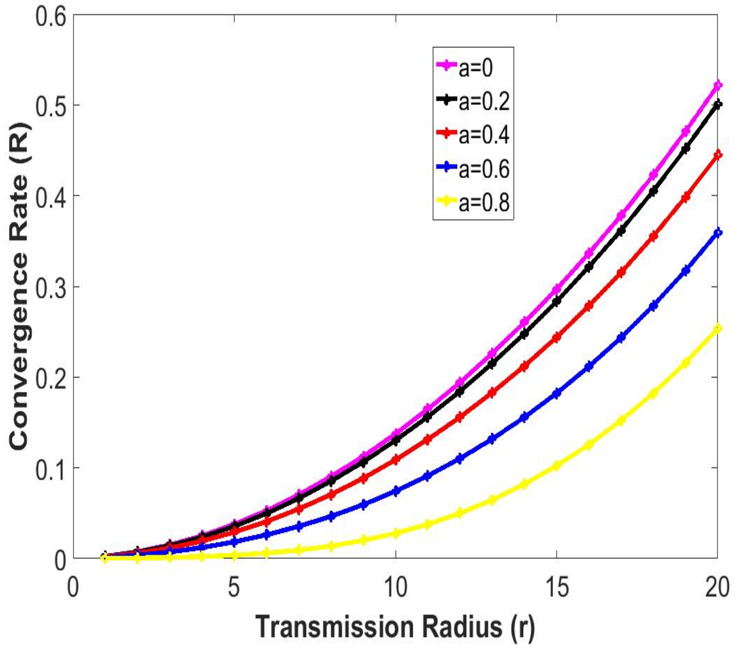

Figure 4.5: Convergence rate versus Asymmetric Link Factor of a r-nearest neighbor network for n=400.

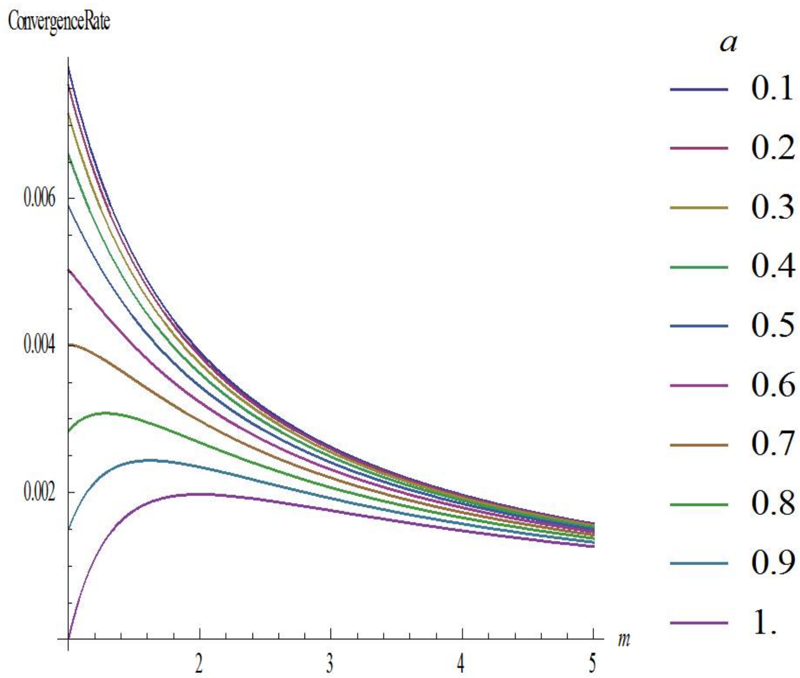

Figure 4.6: Convergence rate versus Network Dimension for m-dimensional torus network.

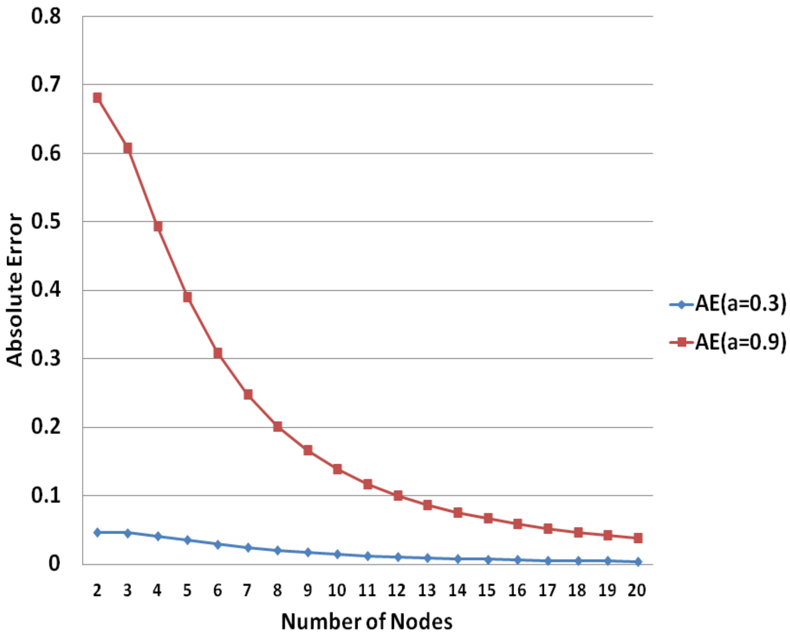

Figure 4.7: Absolute Error versus Number of nodes for $a = 0.3$, and $a = 0.9$.

Chapter 5

Closed-Form Expressions of Convergence rate for One-Dimensional Lattice Networks

5.1 Introduction

Gossip algorithms are considered as an asynchronous version of the consensus algorithms [31], [32], [33], [34], [35], [36], [37], [38]. These algorithms have faster convergence rates with the use of periodic gossip sequences, and such algorithms are termed periodic gossip algorithms [32], [39]. Convergence rate of a periodic gossip algorithm is characterized by the magnitude of the second largest eigenvalue of a gossip matrix [35]. However, computing the second largest eigenvalue requires huge computational resources for large-scale networks. In [35], authors derived the explicit expressions of convergence rate for a ring network of the average periodic gossip algorithms. In this chapter, the convergence rate of the periodic gossip algorithms for one-dimensional lattice network has been derived. WSNs can be modeled by the lattice networks . Lattice networks represent the notion of geographical proximity in the practical WSNs, and they have been extensively used in the WSN applications for measuring and monitoring purposes

[63], [64], [65], [61], [66]. Lattice networks facilitate the closed-form solutions which can be generalized to higher dimensions. These structures also play a fundamental role in analyzing the connectivity, scalability, network size, and node failures in WSNs.
In [31], authors proposed a framework for distributed averaging problem using arbitrary networks. In [32], authors discussed the convergence rate of gossip algorithms and surveyed the recent works of gossip algorithms for WSNs. A finite distributive algorithm is proposed in [34] for ring networks of agents with gossip constraint. In this chapter, we model the WSN as a one-dimensional lattice network and obtain the explicit formulas of convergence rate [3] for the periodic gossip algorithms by considering both even and the odd number of nodes. This work avoids the use of computationally expensive algorithms for studying the large-scale wireless sensor networks.

Organization

This chapter is organized as follows. In section 5.2, closed-form expressions of convergence rate have been derived for average periodic gossip algorithms. To examine the effect of gossip weight on convergence rate, we used the linear weight updated approach and derived the closed-form expressions of convergence rate in section 5.3. In section 5.4, we considered the case of communication link failures and derived the explicit formulas of convergence rate for average periodic gossip algorithms. In section 5.5, we demonstrated the numerical results using MATLAB. Finally, we discuss the conclusions of this chapter in section 5.6.

[3] S. Kouachi, **Sateeshkrishna Dhuli**, and Yatindra. Nath. Singh, "Convergence Rate Analysis for Periodic Gossip Algorithms in Wireless Sensor Networks," *arXiv preprint arXiv:1806.03932* , 2018.

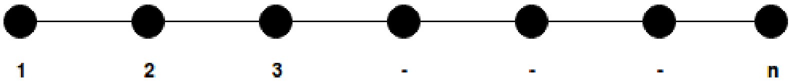

Figure 5.1: One-Dimensional lattice Network

5.2 Average Periodic Gossip Algorithm for a One-Dimensional Lattice Network

As shown in the Fig. 5.1, WSN is modeled as a one-dimensional lattice network. We obtain the optimal periodic sub-sequences to evaluate the primitive gossip matrices. In this algorithm, each pair of nodes at each iteration participate in the gossip process to update with the average of their previous state values to obtain the global average.

Theorem 5.2.1. *Convergence rate of a periodic average gossip algorithm for an one-dimensional lattice network for n=even is expressed as*

$$R = 1 - \sin^2 \tfrac{(n-2)\pi}{2n}. \tag{5.2.1}$$

Proof. The possible pairs for one-dimensional lattice network can be expressed as

$$\{(1,2)(2,3)(3,4)...........(n-2, n-1), (n-1, n)\}$$

In this case, the chromatic index is either 2 or 3. Hence, optimal periodic sub-sequence (E) can be written as

$$E = E_1 E_2,$$

where, $E_1 = \{(2,3)(4,5).....(n-2, n-1)\}$ and
$E_2 = \{(1,2)(3,4)(5,6)..........(n-1, n)\}$ are two disjoint sets.
Primitive gossip matrix (W) is expressed as

$W = S_1S_2$ or $W = S_2S_1$, where

$S_1 = P_{2,3}P_{4,5}......P_{(n-1),n}$,

$S_2 = P_{1,2}P_{3,4}P_{5,6}....P_{(n-2),(n-1)}$.

Hence, gossip matrix (W) for n=even can be computed as

$$W = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{4} & \ddots & 0 & \cdots & \vdots \\ 0 & \frac{1}{4} & \frac{1}{4} & \ddots & \frac{1}{4} & \vdots \\ \vdots & \frac{1}{4} & \ddots & \frac{1}{4} & \frac{1}{4} & 0 \\ \vdots & \cdots & 0 & \ddots & \frac{1}{4} & \frac{1}{2} \\ 0 & \cdots & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \tag{5.2.2}$$

The above matrix structure is in perturbed pentadiagonal format.

Definition 5.2.1. *The eigenvalues of the matrices [67]*

$$A_{2m} = \begin{pmatrix} e-\alpha & b & c & 0 & \ldots & \ldots & \ldots & 0 \\ d & e & b & 0 & \ldots & \ldots & \ldots & \vdots \\ 0 & b & e & b & \ddots & 0 & \ldots & \vdots \\ \vdots & c & b & e & b & \ddots & \ldots & \vdots \\ \vdots & \ldots & \ddots & b & e & b & c & \vdots \\ \vdots & \ldots & 0 & \ddots & b & e & b & 0 \\ \vdots & \ldots & \ldots & \ldots & 0 & b & e & d \\ 0 & \ldots & \ldots & \ldots & 0 & c & b & e-\beta \end{pmatrix} \tag{5.2.3}$$

are the couples $\lambda_{i,k} = e - Y_{i,k},\ i = 1,2$ *and* $k = 0,1,...,m-1$, *where*

$$\begin{cases} Y_{1,0} = c, \quad Y_{2,0} = -2b - c, \\ Y_{i,k}^2 - 2\left(cY_{i,k} - b^2\right)\cos\frac{k\pi}{m} + \left(2b^2 - c^2\right) = 0, \quad k = 1,2,...,m-1. \end{cases} \tag{5.2.4}$$

From (5.2.4), we can write the

$$\lambda_{i,k}^2 - 2e\lambda_{i,k} + e^2 - 2cecos\left(\frac{k\pi}{m}\right) + 2c\lambda_{i,k}cos\left(\frac{k\pi}{m}\right) + 2b^2cos\left(\frac{k\pi}{m}\right) - 2b^2 + c^2 = 0 \quad (5.2.5)$$

Comparing the (5.2.2) and (5.2.3), we can observe that $\alpha = \frac{-1}{4}$, $\beta = \frac{-1}{4}$, $c = \frac{1}{4}$, $e = \frac{1}{4}$, and $b = \frac{1}{4}$. Therefore, (5.2.10) can be rewritten as

$$\lambda_{i,k} = sin^2\frac{k\pi}{n} \quad (5.2.6)$$

when $k = \frac{n}{2}$, $\lambda_{i,k} = 1$. Then, second largest eigenvalue can be obtained at $\frac{n-2}{2}$. Therefore, second largest eigenvalue is expressed as

$$\lambda_{i,\frac{n-2}{2}} = sin^2\frac{(n-2)\pi}{2n} \quad (5.2.7)$$

Substituting (5.2.7) in (2.6.4) completes the proof. ■

Theorem 5.2.2. *Convergence rate of a periodic average gossip algorithm for an one-dimensional lattice network for n=odd is expressed as*

$$R = 1 - \sin^2 \tfrac{(n-2)\pi}{2n}. \quad (5.2.8)$$

Proof. In this case, optimal periodic sub-sequence (E) is expressed as

$$E = E_1E_2, \quad (5.2.9)$$

where, E_1=$\{(2,3)(4,5).....(n-2,n-1)\}$ and

E_2=$\{(1,2)(3,4)(5,6)..........(n-1,n)\}$ are two disjoint sets.

Gossip matrix (W) for n=odd is defined as

$W = S_1S_2$ or $W = S_2S_1$, where

$S_1 = P_{1,2}P_{3,4}......P_{(n-1),n}$

$S_2 = P_{2,3}P_{4,5}P_{6,7}....P_{(n-2),(n-1)}$ Hence, gossip matrix (W) for n=odd can be computed as

$$W = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{4} & \ddots & 0 & \cdots & \vdots \\ 0 & \frac{1}{4} & \frac{1}{4} & \ddots & \frac{1}{4} & \vdots \\ \vdots & \frac{1}{4} & \ddots & \frac{1}{4} & \frac{1}{4} & 0 \\ \vdots & \cdots & 0 & \ddots & \frac{1}{4} & \frac{1}{4} \\ 0 & \cdots & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{5.2.10}$$

Definition 5.2.2. *The eigenvalues of the matrices [67]*

$$A_{2m+1} = \begin{pmatrix} e-\alpha & b & c & 0 & \dots & \dots & \dots & 0 & 0 \\ d & e & b & 0 & \dots & \dots & \dots & \vdots & \vdots \\ 0 & b & e & b & \ddots & 0 & \dots & \vdots & \vdots \\ \vdots & c & b & e & b & \ddots & \dots & \vdots & \vdots \\ \vdots & \dots & \ddots & b & e & b & c & \vdots & \vdots \\ \vdots & \dots & 0 & \ddots & b & e & b & 0 & 0 \\ \vdots & \dots & \dots & \dots & 0 & b & e & b & c \\ 0 & \dots & \dots & \dots & 0 & c & b & e & b \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & d & e-\beta \end{pmatrix} \tag{5.2.11}$$

are the couples $\lambda_{i,k} = e - Y_{i,k},\ \ i = 1,2$ *and* $k = 0,1,...,m-1,$ *where*

$$\begin{cases} Y_{1,0} = -\left(2b + c\right), \quad Y_{2,0} = \frac{b^2}{c}, \\ Y_{i,k}^2 - 2\left(cY_{i,k} - b^2\right)\cos\frac{(2k+1)\pi}{2m+1} + \left(2b^2 - c^2\right) = 0, \quad k = 1,2,...,m-1. \end{cases} \tag{5.2.12}$$

From (5.2.12), we can write

$$\lambda_{i,k}^2 - 2e\lambda_{i,k} + e^2 - 2cecos\left(\frac{(2k+1)\pi}{2m+1}\right) + 2c\lambda_{i,k}cos\left(\frac{(2k+1)\pi}{2m+1}\right) + 2b^2cos\left(\frac{(2k+1)\pi}{2m+1}\right) - 2b^2 + c^2 = 0 \tag{5.2.13}$$

Comparing (5.2.11) and (5.2.12), we can observe that $\alpha = \frac{-1}{4}$, $\beta = \frac{-1}{4}$, $c = \frac{1}{4}$, $e = \frac{1}{4}$, $d = \frac{1}{2}$ and $b = \frac{1}{4}$. Therefore, (5.2.10) can be rewritten as

$$\lambda_{i,k} = sin^2\frac{(2k+1)\pi}{n} \tag{5.2.14}$$

when $k = \frac{n-1}{2}$, $\lambda_{i,k} = 1$. Then, second largest eigenvalue can be obtained at $\frac{n-3}{2}$. Therefore, second largest eigenvalue of weight matrix can be expressed as

$$\lambda_{i,\frac{n-3}{2}} = sin^2\frac{(n-2)\pi}{2n} \tag{5.2.15}$$

Substituting (5.2.15) in (2.6.4) proves the $Theorem$. ■

5.3 Effect of Gossip Weight on Convergence rate

In the previous section, we obtain the primitive gossip matrices for gossip weight $w=\frac{1}{2}$. To investigate the effect of gossip weight on convergence rate, we consider the special case by considering the weights associated with the edges. If we assume that at iteration k, nodes i and j communicate, then node i and node j performs the linear update with gossip weight 'w' as [34], [38].

$$x_i(k) = (1-w)x_i(k-1) + wx_j(k-1), \tag{5.3.1}$$

and

$$x_j(k) = wx_i(k-1) + (1-w)x_j(k-1), \tag{5.3.2}$$

where 'w' is the gossip weight associated with edge (i,j).

Theorem 5.3.1. *Convergence rate of a periodic gossip algorithm for an one-dimensional lattice network with gossip weight 'w' for even number of nodes is expressed as*

$$R = \left|2w - 2w^2\sin^2\tfrac{(n-2)\pi}{2n} - 2w\left|\sin\tfrac{(n-2)\pi}{2n}\right|\sqrt{w^2\sin^2\tfrac{(n-2)\pi}{2n} - 2w + 1}\right|. \tag{5.3.3}$$

Proof. The primitive gossip matrix (W) for n=even can be computed as

$$W = \begin{pmatrix} 1-w & -w^2+w & w^2 & 0 & \cdots & 0 \\ w & (w-1)^2 & \ddots & 0 & \cdots & \vdots \\ 0 & -w^2+w & (w-1)^2 & \ddots & w^2 & \vdots \\ \vdots & w^2 & \ddots & (w-1)^2 & -w^2+w & 0 \\ \vdots & \cdots & 0 & \ddots & (w-1)^2 & w \\ 0 & \cdots & 0 & w^2 & -w^2+w & 1-w \end{pmatrix}. \tag{5.3.4}$$

From (5.3.4), The structure of the above matrix can be expressed in terms of perturbed pentadiagonal matrix. Comparing (5.3.4) and (5.2.3), we observe that $c = w^2$, $\alpha = \beta = w^2 - w$, $d = w$, $e = (w-1)^2$, $b = -w^2 + w$. Therefore, using (5.2.4) we can write

$$\lambda^2 - 2\left[-2w + 1 + 2w^2 \sin^2 \tfrac{k\pi}{n}\right]\lambda + (2w-1)^2 = 0. \tag{5.3.5}$$

Solving (5.3.5) gives the eigenvalues as

$$\begin{aligned} \lambda_k &= -2w + 1 + 2w^2 \sin^2 \tfrac{k\pi}{n} \\ &\pm 2w \left|\sin \tfrac{k\pi}{n}\right| \sqrt{w^2 \sin^2 \tfrac{k\pi}{n} - 2w + 1}, \end{aligned}$$

$k = 1, 2, ..., \frac{n-2}{2}$. The largest eigenvalue is $\lambda_{2,0} = 1$. Consequently, we obtain the second largest eigenvalue at $k = \frac{n-2}{2}$ as

$$\begin{aligned} \lambda^+_{\frac{n-2}{2}} &= -2w + 1 + 2w^2 \sin^2 \tfrac{(n-2)\pi}{2n} \\ &+ 2w \left|\sin \tfrac{(n-2)\pi}{2n}\right| \sqrt{w^2 \sin^2 \tfrac{(n-2)\pi}{2n} - 2w + 1}. \end{aligned} \tag{5.3.6}$$

Substituting (5.3.6) in (2.6.4) proves the $Theorem$. ■

Theorem 5.3.2. *Convergence rate of a periodic gossip algorithm for a one-dimensional lattice network for n=odd is expressed as*

$$R = \left|2w - 2w^2 \sin^2 \tfrac{(n-2)\pi}{2n} - 2w \left|\sin \tfrac{(n-2)\pi}{2n}\right| \sqrt{w^2 \sin^2 \tfrac{(n-2)\pi}{2n} - 2w + 1}\right|. \tag{5.3.7}$$

Proof. For n=*odd*, gossip matrix (W) can be computed as

$$W = \begin{pmatrix} 1-w & -w(w-1) & w^2 & 0 & \cdots & 0 \\ w & (w-1)^2 & \ddots & 0 & \cdots & \vdots \\ 0 & -w(w-1) & (w-1)^2 & \ddots & w^2 & \vdots \\ \vdots & w^2 & \ddots & (w-1)^2 & -w^2+w & 0 \\ \vdots & \cdots & 0 & \ddots & (w-1)^2 & -w(w-1) \\ 0 & \cdots & 0 & 0 & w & (1-w) \end{pmatrix} \tag{5.3.8}$$

Comparing the expressions of (5.3.8) and (5.2.11), we can observe that $c = w^2$, $d = w$, $b = -w^2 + w$, $e = (w-1)^2$, $\alpha = \beta = (w-1)^2 - 1 + w = w^2 - w = -b$. Since $e = (w-1)^2$ and $Y = e - \lambda$, then

$$\lambda_{1,0} = 1.$$

The other eigenvalues are the roots of the quadratic equation

$$\lambda^2 - 2\left[(w-1)^2 - w^2 \cos\tfrac{(2k+1)\pi}{2n}\right]\lambda + (2w-1)^2 = 0,$$

$k = 0, 1, 2, ..., \frac{n-3}{2}$, which can be written

$$\lambda^2 - 2\left[-2w + 1 + 2w^2 \sin^2\tfrac{(2k+1)\pi}{2n}\right]\lambda + (2w-1)^2 = 0, \tag{5.3.9}$$

$k = 0, 1, 2, ..., \frac{n-3}{2}$. Solving (5.3.9) gives the eigenvalues as

$$\begin{aligned}\lambda_k &= -2w + 1 + 2w^2 \sin^2\tfrac{(2k+1)\pi}{2n} \\ &\pm 2w\left|\sin\tfrac{(2k+1)\pi}{2n}\right|\sqrt{w^2 \sin^2\tfrac{(2k+1)\pi}{2n} - 2w + 1}.\end{aligned}$$

The second largest eigenvalue is obtained for $k = \frac{n-3}{2}$. That is

$$\begin{aligned}\lambda^+_{\frac{n-2}{2}} &= -2w + 1 + 2w^2 \sin^2\tfrac{(n-2)\pi}{2n} \\ &+ 2w\left|\sin\tfrac{(n-2)\pi}{2n}\right|\sqrt{w^2 \sin^2\tfrac{(n-2)\pi}{2n} - 2w + 1}.\end{aligned} \tag{5.3.10}$$

Substituting the (5.3.10) in (2.6.4) proves the *Theorem*. ∎

5.4 Effect of Communication Link Failures on Convergence rate

Wireless sensor networks are prone to link failures due to noise, interference, and environmental changes. In this section, we study the effect of link failures on convergence rate for average periodic gossip algorithms. Let us consider the one-dimensional lattice network, where each link fails with the probability 'p'.

Theorem 5.4.1. *If a communication link between two nodes fails with the probability 'p' then convergence rate of the average periodic gossip algorithm for n=even is expressed as*

$$R = 1 - \left| p + \frac{(p-1)^2}{2}\sin^2\frac{(n-2)\pi}{2n} + \sqrt{\frac{(p-1)^4}{4}\sin^4\frac{(n-2)\pi}{2n} + p(p-1)^2\sin^2\frac{(n-2)\pi}{2n}} \right|. \quad (5.4.1)$$

Proof. Let us consider the one-dimensional lattice network, where each link fails with the probability 'p'. Then, probability that two nodes connected by a communication link is '*1-p*'. Primitive gossip matrix for even number of nodes is expressed as

$$W = \begin{pmatrix} \frac{p+1}{2} & \frac{1-p^2}{4} & \frac{(1-p)^2}{4} & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{1-p}{2} & \frac{(p+1)^2}{4} & \frac{1-p^2}{2} & 0 & \cdots & \cdots & \cdots & \vdots \\ 0 & \frac{1-p^2}{4} & \frac{(p+1)^2}{4} & \frac{1-p^2}{4} & \ddots & 0 & \cdots & \vdots \\ \vdots & \frac{(1-p)^2}{4} & \frac{1-p^2}{4} & \frac{(p+1)^2}{4} & \frac{1-p^2}{4} & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \frac{1-p^2}{4} & \frac{(p+1)^2}{4} & \frac{1-p^2}{4} & \frac{(1-p)^2}{4} & \vdots \\ \vdots & \cdots & 0 & \ddots & \frac{1-p^2}{4} & \frac{(p+1)^2}{4} & \frac{1-p^2}{4} & 0 \\ \vdots & \cdots & \cdots & \cdots & 0 & \frac{1-p^2}{4} & \frac{(p+1)^2}{4} & \frac{1-p}{2} \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{(p-1)^2}{4} & \frac{1-p^2}{4} & \frac{p+1}{2} \end{pmatrix}. \quad (5.4.2)$$

Comparing the expressions (5.4.2) and (5.2.3), we observe that $c = \frac{(p-1)^2}{4}$, $d = \frac{1-p}{2}$, $b = \frac{1-p^2}{4}$, $e = \frac{(p+1)^2}{4}$, $\alpha = \beta = \frac{p^2-1}{4}$, and $Y = e - \lambda$, then $\lambda_{1,0} = 1$.

The other eigenvalues are the roots of the quadratic equation

$$\lambda^2 + \lambda\left(-2e + 2c\cos\tfrac{k\pi}{m}\right) + e^2 - 2ce\cos\tfrac{k\pi}{m} + 2b^2\cos\tfrac{k\pi}{m} - 2b^2\cos\tfrac{k\pi}{m} - 2b^2 + c^2 = 0 \tag{5.4.3}$$

Substituting the values of c, b, and e results in

$$\lambda^2 + \lambda\left(-\tfrac{(p+1)^2}{2} + \tfrac{(p-1)^2}{2} - (p-1)^2\sin^2\tfrac{k\pi}{n}\right) + p^2 = 0 \tag{5.4.4}$$

Solving (5.4.3), gives the expressions for eigenvalues

$$\lambda_k = p + \tfrac{(p-1)^2}{2}\sin^2\tfrac{k\pi}{n} + \sqrt{\tfrac{(p-1)^4}{4}\sin^4\tfrac{k\pi}{n} + p(p-1)^2\sin^2\tfrac{k\pi}{n}} \tag{5.4.5}$$

Then, second largest eigenvalue is expressed as

$$\lambda^+_{\frac{n-3}{2}} = p + \tfrac{(p-1)^2}{2}\sin^2\tfrac{(n-2)\pi}{2n} + \sqrt{\tfrac{(p-1)^4}{4}\sin^4\tfrac{(n-2)\pi}{2n} + p(p-1)^2\sin^2\tfrac{(n-2)\pi}{2n}} \tag{5.4.6}$$

Substituting the (5.4.6) in (2.6.4) proves the *Theorem*. ■

Theorem 5.4.2. *If a communication link between two nodes fails with the probability p, then convergence rate of the average periodic gossip algorithm for n=odd is expressed as*

$$R = 1 - \left|p + \tfrac{(p-1)^2}{2}\sin^2\tfrac{(n-2)\pi}{2n} + \sqrt{\tfrac{(p-1)^4}{4}\sin^4\tfrac{(n-2)\pi}{2n} + p(p-1)^2\sin^2\tfrac{(n-2)\pi}{2n}}\right|. \tag{5.4.7}$$

Proof. Primitive gossip matrix for odd number of nodes when communication links between any two nodes fails with the probability p is expressed as

$$W = \begin{pmatrix} \frac{1+p}{2} & \frac{1-p^2}{4} & \frac{(p-1)^2}{4} & 0 & \dots & \dots & \dots & 0 & 0 \\ \frac{1-p}{2} & \frac{(p+1)^2}{4} & \frac{1-p^2}{4} & 0 & \dots & \dots & \dots & \vdots & \vdots \\ 0 & \frac{1-p^2}{4} & \frac{(p+1)^2}{4} & \frac{1-p^2}{4} & \ddots & 0 & \dots & \vdots & \vdots \\ \vdots & \frac{(p-1)^2}{4} & \frac{1-p^2}{4} & \frac{(p+1)^2}{4} & \frac{1-p^2}{4} & \ddots & \dots & \vdots & \vdots \\ \vdots & \dots & \ddots & \frac{1-p^2}{4} & \frac{(p+1)^2}{4} & \frac{1-p^2}{4} & \frac{(p-1)^2}{4} & \vdots & \vdots \\ \vdots & \dots & 0 & \ddots & \frac{1-p^2}{4} & \frac{(p+1)^2}{4} & \frac{(1-p^2)}{4} & 0 & 0 \\ \vdots & \dots & \dots & \dots & 0 & \frac{1-p^2}{4} & \frac{(p+1)^2}{4} & \frac{1-p^2}{4} & \frac{(p-1)^2}{4} \\ 0 & \dots & \dots & \dots & 0 & \frac{(p-1)^2}{4} & \frac{1-p^2}{4} & \frac{(p+1)^2}{4} & \frac{1-p^2}{4} \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & \frac{1-p}{2} & \frac{1+p}{2} \end{pmatrix}. \tag{5.4.8}$$

Comparing the expressions (5.4.3) and (5.2.11), we can observe that $c = \frac{(p-1)^2}{4}$, $d = \frac{1-p}{2}$, $b = \frac{1-p^2}{4}$, $e = \frac{(p+1)^2}{4}$, $\alpha = \beta = \frac{p^2-1}{4}$ and $Y = e - \lambda$, then $\lambda_{1,0} = 1$. The other eigenvalues are the roots of the quadratic equation

$$\lambda^2 + \lambda\left(-2e + 2c\cos\frac{(2k+1)\pi}{2m+1}\right) + e^2 - 2ce\cos\frac{(2k+1)\pi}{2m+1} + 2b^2\cos\frac{(2k+1)\pi}{2m+1} - 2b^2 + c^2 = 0 \tag{5.4.9}$$

Substituting the values of c, b, and e results in

$$\lambda^2 + \lambda\left(-\frac{(p+1)^2}{2} + \frac{(p-1)^2}{2} - (p-1)^2\sin^2\frac{(2k+1)\pi}{4m+2}\right) + p^2 = 0 \tag{5.4.10}$$

Solving (5.4.5) gives the eigenvalues as

$$\lambda_k = p + \frac{(p-1)^2}{2}\sin^2\frac{(2k+1)\pi}{4m+2} + \sqrt{\frac{(p-1)^4}{4}\sin^4\frac{(2k+1)\pi}{4m+2} + p(p-1)^2\sin^2\frac{(2k+1)\pi}{4m+2}} \tag{5.4.11}$$

Then, second largest eigenvalue is expressed as

$$\lambda^+_{\frac{n-2}{2}} = p + \frac{(p-1)^2}{2}\sin^2\frac{(n-2)\pi}{2n} + \sqrt{\frac{(p-1)^4}{4}\sin^4\frac{(n-2)\pi}{2n} + p(p-1)^2\sin^2\frac{(n-2)\pi}{2n}} \tag{5.4.12}$$

Substituting the (5.4.12) in (2.6.4) proves the *Theorem*. ■

Table 5.1: Convergence rate of Small-Scale WSNs

Number of Nodes	**Convergence rate**	**Optimal Gossip Weight**
4	0.8	0.6
5	0.6	0.7
6	0.6	0.7
7	0.6	0.7
8	0.4	0.8
9	0.4	0.8
10	0.4	0.8
11	0.4	0.8
12	0.4	0.8
13	0.3034	0.8
14	0.2412	0.8
15	0.2015	0.8
16	0.2	0.9
17	0.2	0.9
18	0.2	0.9
19	0.2	0.9
20	0.2	0.9

Table 5.2: Convergence rate of Large-Scale WSNs

Number of Nodes	**Convergence rate**	**Optimal Gossip Weight**
100	0.009	0.9
200	0.0022	0.9
300	0.001	0.9
400	0.0006	0.9
500	0.1	0.9
600	0.002	0.9
700	0.002	0.9
800	0.001	0.9
900	0.001	0.9
1000	0.0001	0.9

5.5 Numerical Results and Discussion

In this section, numerical results have been presented. Fig. 5.2 shows the convergence rate versus the number of nodes in one-dimensional lattice networks for average periodic gossip algorithms (w=0.5). It has been observed that the convergence rate reduces exponentially with the increase in the number of nodes. In every time step, nodes share information with their direct neighbors to achieve the global average. Thus, for the larger number of nodes, more time steps will be required, thereby leading to slower convergence rates. As shown in Table. 5.1, optimal gossip weights are varying with the number of nodes until n=16 and it's value becomes 0.9 from $n \geq 16$. Fig. 5.3 shows the convergence rate versus gossip weights for large-scale networks. It has been observed that for large-scale lattice networks, the optimal gossip weight turns out to be 0.9 (see Table. 5.2). Hence, it can be concluded that for any medium-scale network ($n \geq 16$), gossip weight should be 0.9 for achieving faster convergence rates in one-dimensional lattice networks. It has been measured that the efficiency of periodic gossip algorithms at w=0.9 over average periodic gossip algorithms(w=0.5) by using relative error ($E = \frac{R_{0.9} - R_{0.5}}{R_{0.9}}$) where $R_{0.9}$ and $R_{0.5}$ denote the convergence rate for w=0.9 and w=0.5 respectively. Fig. 5.5 shows the relative error versus the number of nodes for periodic gossip algorithms. Here, we have observed that relative error increases with the network size in small-scale networks and it becomes unity for large-scale networks. To study the effect of communication link failures, Fig. 5.6 has been plotted. We have observed that the convergence rate decreases with the probability of link failures. As shown in the Fig. 5.6, convergence rate becomes zero for p=1.

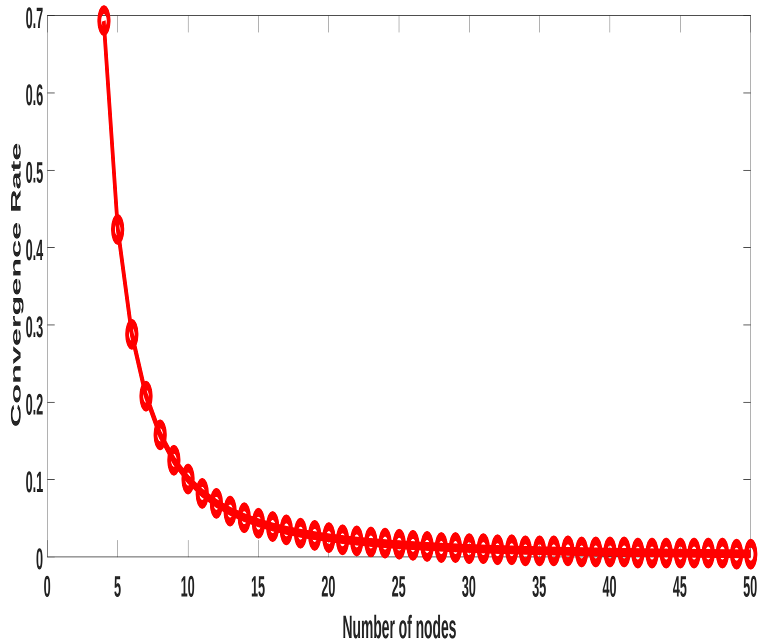

Figure 5.2: Convergence rate versus Number of nodes for Average Gossip Algorithm (w=0.5).

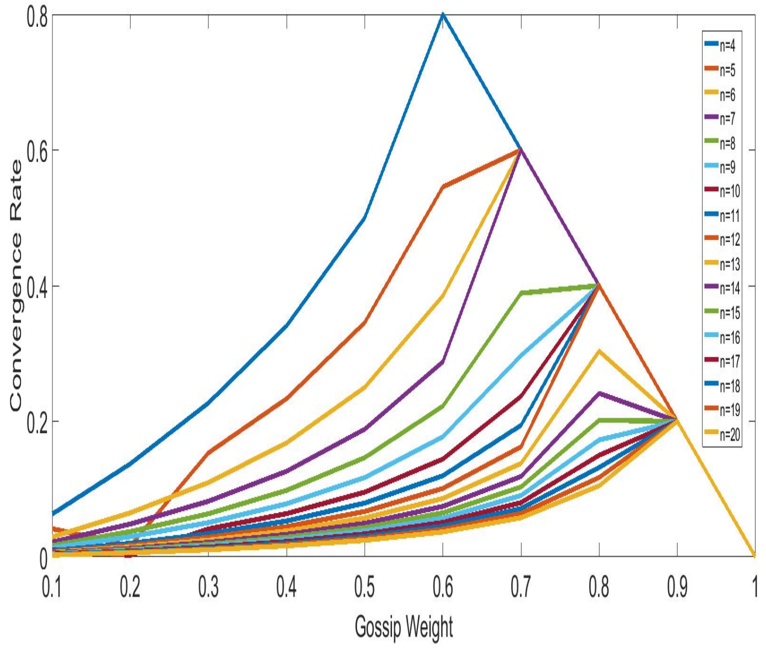

Figure 5.3: Convergence rate versus Gossip Weight for Small Scale WSNs.

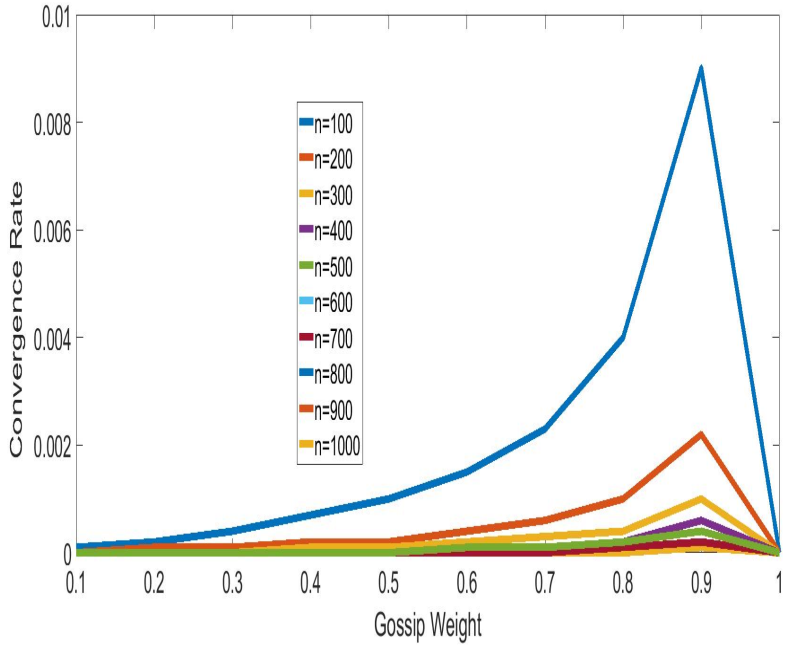

Figure 5.4: Convergence rate versus Gossip Weight for Large Scale WSNs.

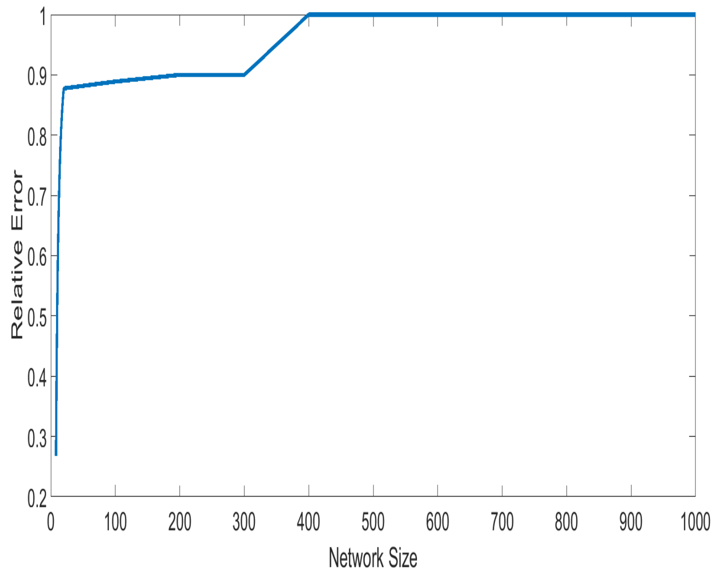

Figure 5.5: Relative Error versus Network Size.

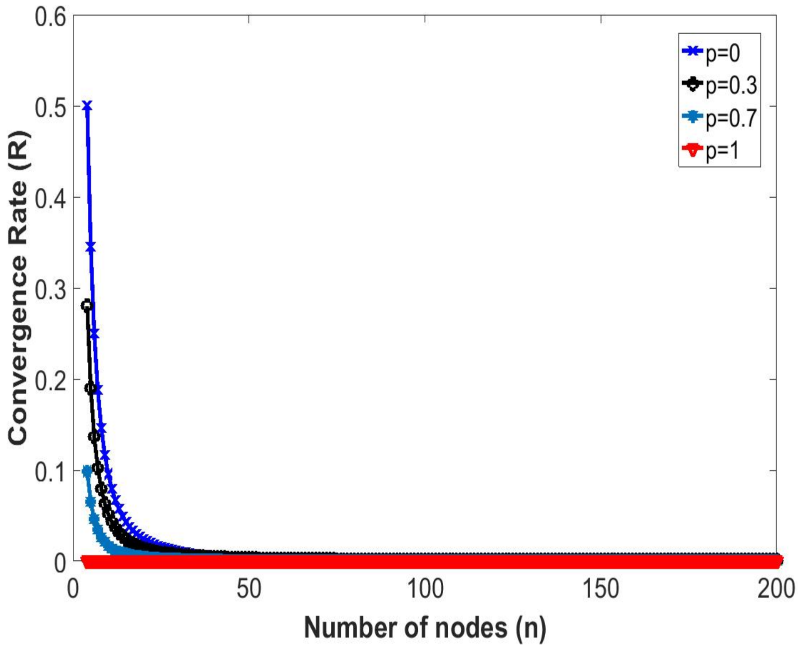

Figure 5.6: Effect of Link Failures on Convergence rate.

5.6 Conclusions

Estimating the convergence rate of a periodic gossip algorithm is computationally challenging in large-scale networks. This chapter derived the explicit formulas of convergence rate for one-dimensional lattice networks. This work drastically reduces the computational complexity to estimate the convergence rate for large-scale WSNs. Explicit expressions have been derived for convergence rate in terms of gossip weight and the number of nodes using the linear weight updating approach. Based on the findings, it has been observed that there exists an optimum gossip weight which significantly improves the convergence rate for periodic gossip algorithms in small-scale WSNs ($n < 16$). Numerical results demonstrated that periodic gossip algorithms achieve faster convergence rate for large-scale networks ($n \geq 16$) at w=0.9 over average periodic gossip algorithms. In this chapter, communication link failures in WSNs has also been considered, and the closed-form expressions of convergence rate have been derived for average periodic gossip algorithms.

Chapter 6

Analysis of Periodic Gossip Algorithms for r-Nearest Neighbor Networks

6.1 Introduction

In this chapter, WSN has been modeled as an r-nearest neighbor network and the effect of transmission radius on convergence rate has been studied for periodic gossip algorithms. Periodic gossip algorithms received much attention in the recent times for achieving faster convergence rate over gossip algorithms [31], [32], [33], [34], [35], [36], [37], [38]. They are quite suitable for data delivery in WSNs as they can be utilized when the global network topology is highly dynamic, and the network consists of power constrained nodes [21], [39]. Ring networks, r-nearest neighbor networks have been extensively used in the literature to study the properties of WSNs [49], [68]. In [35], authors derived the explicit expressions of convergence rate for a ring network of the average periodic gossip algorithms. Closed-form expressions of convergence rate for one-dimensional lattice networks have been derived in [67]. In this chapter, WSN has been modeled as an r-nearest neighbor ring network and study the convergence rate of average periodic gossip algorithms. The variable r models the nodes transmission radius or overhead in WSNs [49], [68]. The combination of power-iteration and deflation techniques has been proposed to evaluate the convergence rate of r-nearest neighbor ring networks. Further, the impact of node's transmission radius on the convergence rate of average periodic gossip algorithms has been studied for both even and the odd number of nodes.

r-Nearest Neighbor Ring Networks

As shown in the Fig. 6.2, every node connected to 2-hop away neighbors in a 2-nearest neighbor ring network. Similarly, in a 4-nearest neighbor ring network, every node gets connected to all the nodes with in 4-hops away in a ring network.

Organization

This chapter is organized as follows. In section 6.2, we obtain the optimal periodic gossip sequences and compute the primitive gossip matrices for n=9 varying r from 1 to 4. In section 6.3, we obtain the optimal periodic gossip sequences and compute the primitive gossip matrices for n=10 varying r from 1 to 4. To evaluate the convergence rate, we propose the combination of power-iteration and deflation techniques in section 6.4. Numerical results have been presented in section 6.5. Finally, we discuss the conclusions of this chapter in section 6.6.

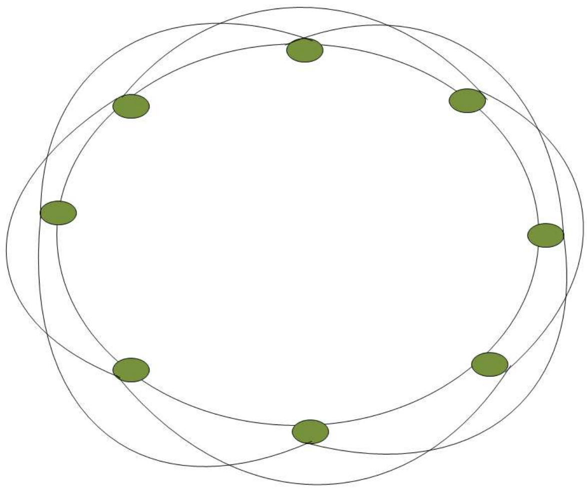

Figure 6.1: 2-nearest neighbor ring network.

6.2 Periodic Sequences of 4-Nearest Neighbor Network for odd number of nodes

Nearest Neighbors (r=1)

The possible pairs of a 1-nearest neighbor network for n=9 can be expressed as

$$(1,2)(2,3)(3,4)(4,5)(5,6)(6,7)(7,8)(8,9)(9,1) \tag{6.2.1}$$

In this case, the chromatic index is either 2 or 3. Hence, optimal periodic sub-sequence (E) can be written as

$$E = E_1 E_2 E_3,$$

where,

$$\begin{aligned} E_1 &= (1,2)(3,4)(5,6)(7,8) \\ E_2 &= (2,3)(4,5)(6,8)(7,9) \\ E_3 &= (1,9) \end{aligned} \tag{6.2.2}$$

are disjoint sets.

Primitive gossip matrix (W) is expressed as

$W = S_1 S_2$, where

$S_1 = P_{1,2}P_{3,4}P_{5,6}P_{7,8}$,

$S_2 = P_{2,3}P_{4,5}P_{6,8}P_{7,9}$.

$S_3 = P_{1,9}$. Hence, W is computed as

$$\begin{pmatrix} 0.25 & 0.25 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0.25 \\ 0 & 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0.25 & 0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.25 & 0.25 & 0.25 & 0 & 0.25 & 0 \\ 0.125 & 0 & 0 & 0 & 0 & 0.25 & 0.25 & 0.25 & 0.125 \\ 0.125 & 0 & 0 & 0 & 0 & 0.25 & 0.25 & 0.25 & 0.125 \\ 0.25 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.25 \end{pmatrix} \tag{6.2.3}$$

Nearest Neighbors (r=2)

The possible pairs of a 2-nearest neighbor network for n=9 can be expressed as

$$\begin{array}{l} (1,2)(2,3)(3,4)(4,5)(5,6)(6,7)(7,8)(8,9)(9,1) \\ (1,3)(2,4)(3,5)(4,6)(5,7)(6,8)(7,9)(8,1)(9,2) \end{array} \tag{6.2.4}$$

In this case, the chromatic index is either 4 or 5. Hence, optimal periodic sub-sequence (E) can be written as

$$E = E_1E_2E_3E_4E_5,$$

$$\begin{array}{l} E_1 = (1,2)(5,6)(7,9)(3,4) \\ E_2 = (2,3)(9,1)(4,5)(6,8) \\ E_3 = (7,8)(4,6)(1,3) \\ E_4 = (9,2)(3,5)(6,7)(8,1) \\ E_5 = (2,4)(8,9)(5,7) \end{array} \tag{6.2.5}$$

are disjoint sets.

Primitive gossip matrix (W) is expressed as

$W = S_1 S_2 S_3 S_4 S_5$, where

$S_1 = P_{1,2} P_{5,6} P_{7,9} P_{3,4}$,

$S_2 = P_{2,3} P_{9,1} P_{4,5} P_{6,8}$,

$S_3 = P_{7,8} P_{4,6} P_{1,3}$,

$S_4 = P_{9,2} P_{3,5} P_{6,7} P_{8,1}$,

$S_5 = P_{2,4} P_{8,9} P_{5,7}$.

Hence, W is computed as

$$\begin{pmatrix} 0.1250 & 0.1250 & 0.1250 & 0.1250 & 0.0625 & 0 & 0.0625 & 0.1875 & 0.1875 \\ 0.1250 & 0.1250 & 0.1250 & 0.1250 & 0.0625 & 0 & 0.0625 & 0.1875 & 0.1875 \\ 0.0625 & 0.1250 & 0.1875 & 0.1250 & 0.1250 & 0.0625 & 0.1250 & 0.0938 & 0.0938 \\ 0.0625 & 0.1250 & 0.1875 & 0.1250 & 0.1250 & 0.0625 & 0.1250 & 0.0938 & 0.0938 \\ 0.0625 & 0.1250 & 0.1250 & 0.1250 & 0.1563 & 0.1875 & 0.1563 & 0.0313 & 0.0313 \\ 0.0625 & 0.1250 & 0.1250 & 0.1250 & 0.1563 & 0.1875 & 0.1563 & 0.0313 & 0.0313 \\ 0.1875 & 0.0625 & 0.0625 & 0.0625 & 0.0938 & 0.1250 & 0.0938 & 0.1563 & 0.1563 \\ 0.1250 & 0.1250 & 0 & 0.1250 & 0.1250 & 0.2500 & 0.1250 & 0.0625 & 0.0625 \\ 0.1875 & 0.0625 & 0.0625 & 0.0625 & 0.0938 & 0.1250 & 0.0938 & 0.1563 & 0.1563 \end{pmatrix} \tag{6.2.6}$$

Nearest Neighbors (r=3)

The possible pairs of a 3-nearest neighbor network for n=9 can be expressed as

$$\begin{array}{l} (1,2)(2,3)(3,4)(4,5)(5,6)(6,7)(7,8)(8,9)(9,1) \\ (1,3)(2,4)(3,5)(4,6)(5,7)(6,8)(7,9)(8,1)(9,2) \\ (1,4)(2,5)(3,6)(4,7)(5,8)(6,9)(7,1)(8,2)(9,3) \end{array} \tag{6.2.7}$$

In this case, the chromatic index is either 6 or 7. Hence, optimal periodic sub-sequence (E) can be written as

$$E = E_1 E_2 E_3 E_4 E_5 E_6 E_7,$$

where

$$\begin{aligned}
E_1 &= (1,2)(9,6)(4,7)(5,8)\\
E_2 &= (2,3)(9,1)(6,8)(4,5)\\
E_3 &= (3,4)(5,7)(2,9)(1,8)\\
E_4 &= (1,3)(7,9)(5,6)(2,4)\\
E_5 &= (6,7)(3,5)(1,4)(8,2)\\
E_6 &= (7,8)(4,6)(2,5)(3,9)\\
E_7 &= (6,3)(7,1)(8,9)
\end{aligned} \tag{6.2.8}$$

are disjoint sets. Primitive gossip matrix (W) is expressed as

$W = S_1S_2S_3S_4S_5S_6S_7$, where

$S_1 = P_{1,2}P_{9,6}P_{4,7}P_{5,8}$,

$S_2 = P_{2,3}P_{9,1}P_{6,8}P_{4,5}$,

$S_3 = P_{3,4}P_{5,7}P_{2,9}P_{1,8}$,

$S_4 = P_{1,3}P_{7,9}P_{5,6}P_{2,4}$,

$S_5 = P_{6,7}P_{3,5}P_{1,4}P_{8,2}$,

$S_6 = P_{7,8}P_{4,6}P_{2,5}P_{3,9}$,

$S_7 = P_{6,3}P_{7,1}P_{8,9}$.

Hence, W is computed as

$$\begin{pmatrix}
0.1328 & 0.1094 & 0.1016 & 0.1094 & 0.1094 & 0.1016 & 0.1328 & 0.1016 & 0.1016\\
0.1328 & 0.1094 & 0.1016 & 0.1094 & 0.1094 & 0.1016 & 0.1328 & 0.1016 & 0.1016\\
0.1406 & 0.0938 & 0.1094 & 0.1250 & 0.0938 & 0.1094 & 0.1406 & 0.0938 & 0.0938\\
0.0859 & 0.0781 & 0.1406 & 0.1250 & 0.0781 & 0.1406 & 0.0859 & 0.1328 & 0.1328\\
0.1016 & 0.1250 & 0.1094 & 0.1094 & 0.1250 & 0.1094 & 0.1016 & 0.1094 & 0.1094\\
0.1094 & 0.1406 & 0.0938 & 0.0938 & 0.1406 & 0.0938 & 0.1094 & 0.1094 & 0.1094\\
0.0859 & 0.0781 & 0.1406 & 0.1250 & 0.0781 & 0.1406 & 0.0859 & 0.1328 & 0.1328\\
0.1016 & 0.1250 & 0.1094 & 0.1094 & 0.1250 & 0.1094 & 0.1016 & 0.1094 & 0.1094\\
0.1094 & 0.1406 & 0.0938 & 0.0938 & 0.1406 & 0.0938 & 0.1094 & 0.1094 & 0.1094
\end{pmatrix} \tag{6.2.9}$$

Nearest Neighbors (r=4)

The possible pairs of a 4-nearest neighbor network for n=9 can be expressed as

$$\begin{array}{l}(1,2)(2,3)(3,4)(4,5)(5,6)(6,7)(7,8)(8,9)(9,1)\\(1,3)(2,4)(3,5)(4,6)(5,7)(6,8)(7,9)(8,1)(9,2)\\(1,4)(2,5)(3,6)(4,7)(5,8)(6,9)(7,1)(8,2)(9,3)\\(1,5)(2,6)(3,7)(4,8)(5,9)(6,1)(7,2)(8,3)(9,4)\end{array} \tag{6.2.10}$$

In this case, the chromatic index is either 8 or 9. Hence, optimal periodic sub-sequence (E) can be written as

$$E = E_1E_2E_3E_4E_5E_6E_7E_8E_9,$$

where

$$\begin{array}{l}E_1 = (1,2)(4,3)(6,5)(8,9)\\E_2 = (4,5)(9,6)(8,2)(7,3)\\E_3 = (1,4)(2,3)(7,5)(8,6)\\E_4 = (1,5)(9,3)(7,6)(2,4)\\E_5 = (1,6)(5,3)(7,8)(9,4)\\E_6 = (1,7)(6,3)(8,5)(9,2)\\E_7 = (1,8)(9,5)(4,6)(7,2)\\E_8 = (1,9)(8,3)(2,5)(4,7)\\E_9 = (7,9)(2,6)(4,8)(1,3)\end{array} \tag{6.2.11}$$

are disjoint sets.

Primitive gossip matrix (W) is expressed as

$W = S_1S_2S_3S_4S_5S_6S_7$, where

$S_1 = P_{1,2}P_{4,3}P_{6,5}P_{8,9}$,

$S_2 = P_{4,5}P_{9,6}P_{8,2}P_{7,3}$,

$S_3 = P_{1,4}P_{2,3}P_{7,5}P_{8,6}$,

$S_4 = P_{1,5}P_{9,3}P_{7,6}P_{2,4}$,

$S_5 = P_{1,6}P_{5,3}P_{7,8}P_{9,4}$,

$S_6 = P_{1,7}P_{6,3}P_{8,5}P_{9,2}$,

$S_7 = P_{1,8}P_{9,5}P_{4,6}P_{7,2}$,

$S_8 = P_{1,9}P_{8,3}P_{2,5}P_{4,7}$,

$S_9 = P_{7,9}P_{2,6}P_{4,8}P_{1,3}$.

Hence, W is computed as

$$\begin{pmatrix} 0.1016 & 0.1172 & 0.1016 & 0.1055 & 0.1250 & 0.1172 & 0.1133 & 0.1055 & 0.1133 \\ 0.1016 & 0.1172 & 0.1016 & 0.1055 & 0.1250 & 0.1172 & 0.1133 & 0.1055 & 0.1133 \\ 0.1113 & 0.1113 & 0.1113 & 0.1172 & 0.1055 & 0.1113 & 0.1074 & 0.1172 & 0.1074 \\ 0.1113 & 0.1113 & 0.1113 & 0.1172 & 0.1055 & 0.1113 & 0.1074 & 0.1172 & 0.1074 \\ 0.1191 & 0.1055 & 0.1191 & 0.1152 & 0.1016 & 0.1055 & 0.1094 & 0.1152 & 0.1094 \\ 0.1191 & 0.1055 & 0.1191 & 0.1152 & 0.1016 & 0.1055 & 0.1094 & 0.1152 & 0.1094 \\ 0.1055 & 0.1172 & 0.1055 & 0.1133 & 0.1094 & 0.1172 & 0.1094 & 0.1133 & 0.1094 \\ 0.1152 & 0.1074 & 0.1152 & 0.1055 & 0.1133 & 0.1074 & 0.1152 & 0.1055 & 0.1152 \\ 0.1152 & 0.1074 & 0.1152 & 0.1055 & 0.1133 & 0.1074 & 0.1152 & 0.1055 & 0.1152 \end{pmatrix} \tag{6.2.12}$$

6.3 Periodic Sequences for 4-Nearest Neighbor Network for even number of nodes

We consider the 4-nearest neighbor ring network. Here, we consider the n=10. In this algorithm, each pair of nodes at each iteration participate in the gossip process to update the average of their previous state values to obtain the global average. The variable r is varied from 1 to 4 and optimal periodic gossip sub-sequences have been computed.

Nearest Neighbors (r=1)

The possible pairs of a 1-nearest neighbor network for $n = 10$ can be expressed as

$$(1,2)(2,3)(3,4)(4,5)(5,6)(6,7)(7,8)(8,9)(9,10)(10,1) \tag{6.3.1}$$

In this case, the chromatic index is either 2 or 3. Hence, optimal periodic sub-sequence (E) can be written as

$$E = E_1 E_2,$$

where,

$$\begin{aligned} E_1 &= (1,2)(3,4)(5,6)(7,8)(9,10) \\ E_2 &= (2,3)(4,5)(6,7)(8,9)(1,10) \end{aligned} \tag{6.3.2}$$

are two disjoint sets.

Primitive gossip matrix (W) is expressed as

$W = S_1 S_2$, where

$S_1 = P_{1,2} P_{3,4} P_{5,6} P_{7,8} P_{9,10}$,

$S_2 = P_{2,3} P_{4,5} P_{6,7} P_{8,9} P_{1,10}$.

Hence, W is computed as

$$\begin{pmatrix} 0.2500 & 0.2500 & 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 \\ 0.2500 & 0.2500 & 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 \\ 0 & 0.2500 & 0.2500 & 0.2500 & 0.2500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2500 & 0.2500 & 0.2500 & 0.2500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2500 & 0.2500 & 0.2500 & 0.2500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2500 & 0.2500 & 0.2500 & 0.2500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2500 & 0.2500 & 0.2500 & 0.2500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2500 & 0.2500 & 0.2500 & 0.2500 & 0 \\ 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 & 0.2500 & 0.2500 \\ 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 & 0.2500 & 0.2500 \end{pmatrix} \tag{6.3.3}$$

Nearest Neighbors (r=2)

The possible pairs of a 2-nearest neighbor network can be expressed as

$$\begin{aligned} &(1,2)(2,3)(3,4)(4,5)(5,6)(6,7)(7,8)(8,9)(9,10)(10,1), \\ &(1,3)(2,4)(3,5)(4,6)(5,7)(6,8)(7,9)(8,10)(9,1)(10,2). \end{aligned} \tag{6.3.4}$$

In this case, the chromatic index is either 4 or 5. Hence, optimal periodic sub-sequence (E) can be written as

$$E = E_1E_2E_3E_4E_5,$$

where,

$$\begin{array}{l} E_1 = (1,2)(3,4)(5,6)(7,8)(9,10) \\ E_2 = (2,3)(4,5)(6,7)(8,9)(10,1) \\ E_3 = (1,3)(2,4)(5,7)(6,8) \\ E_4 = (3,5)(4,6)(7,9)(8,10) \\ E_5 = (9,1)(10,2) \end{array} \tag{6.3.5}$$

are disjoint sets.

Primitive gossip matrix (W) is expressed as

$W = S_1S_2S_3S_4S_5$, where

$S_1 = P_{1,2}P_{3,4}P_{5,6}P_{7,8}P_{9,10}$,

$S_2 = P_{2,3}P_{4,5}P_{6,7}P_{8,9}P_{10,1}$,

$S_3 = P_{1,3}P_{2,4}P_{5,7}P_{6,8}$,

$S_4 = P_{3,5}P_{4,6}P_{7,9}P_{8,10}$,

$S_5 = P_{9,1}P_{10,2}$.

Hence, W is computed as

$$\begin{pmatrix} 0.1250 & 0.1250 & 0.2500 & 0.1250 & 0 & 0 & 0 & 0.1250 & 0.1250 & 0.1250 \\ 0.1250 & 0.1250 & 0.2500 & 0.1250 & 0 & 0 & 0 & 0.1250 & 0.1250 & 0.1250 \\ 0.0938 & 0.1250 & 0.1250 & 0.2500 & 0.1250 & 0 & 0.0625 & 0 & 0.0938 & 0.1250 \\ 0.0938 & 0.1250 & 0.1250 & 0.2500 & 0.1250 & 0 & 0.0625 & 0 & 0.0938 & 0.1250 \\ 0.0625 & 0.0938 & 0 & 0.1250 & 0.2500 & 0.1250 & 0.1250 & 0.0625 & 0.0625 & 0.0938 \\ 0.0625 & 0.0938 & 0 & 0.1250 & 0.2500 & 0.1250 & 0.1250 & 0.0625 & 0.0625 & 0.0938 \\ 0.0938 & 0.0625 & 0 & 0 & 0.1250 & 0.2500 & 0.1875 & 0.1250 & 0.0938 & 0.0625 \\ 0.0938 & 0.0625 & 0 & 0 & 0.1250 & 0.2500 & 0.1875 & 0.1250 & 0.0938 & 0.0625 \\ 0.1250 & 0.0938 & 0.1250 & 0 & 0 & 0.1250 & 0.1250 & 0.1875 & 0.1250 & 0.0938 \\ 0.1250 & 0.0938 & 0.1250 & 0 & 0 & 0.1250 & 0.1250 & 0.1875 & 0.1250 & 0.0938 \end{pmatrix} \tag{6.3.6}$$

Nearest Neighbors (r=3)

The possible pairs of a 3-nearest neighbor network can be expressed as

$$\begin{array}{l}(1,2)(2,3)(3,4)(4,5)(5,6)(6,7)(7,8)(8,9)(9,10)(10,1)\\(1,3)(2,4)(3,5)(4,6)(5,7)(6,8)(7,9)(8,10)(9,1)(10,2)\\(1,4)(2,5)(3,6)(4,7)(5,8)(6,9)(7,10)(8,1)(9,2)(10,3)\end{array} \qquad (6.3.7)$$

In this case, the chromatic index is either 6 or 7. Hence, optimal periodic sub-sequence (E) can be written as

$$E = E_1E_2E_3E_4E_5E_6E_7,$$

where,

$$\begin{array}{l}E_1 = (1,2)(3,4)(5,6)(7,8)(9,10)\\E_2 = (2,3)(4,5)(6,7)(8,9)(10,1)\\E_3 = (1,3)(2,4)(5,7)(6,8)\\E_4 = (3,5)(4,6)(7,9)(8,10)\\E_5 = (9,1)(10,2)(3,6)(5,8)\\E_6 = (1,4)(2,5)(6,9)(7,10)\\E_7 = (4,7)(8,1)(9,2)(10,3)\end{array} \qquad (6.3.8)$$

are disjoint sets.

Primitive gossip matrix (W) is expressed as

$W = S_1S_2S_3S_4S_5S_6S_7$, where

$S_1 = P_{1,2}P_{3,4}P_{5,6}P_{7,8}P_{9,10}$,

$S_2 = P_{2,3}P_{4,5}P_{6,7}P_{8,9}P_{10,1}$,

$S_3 = P_{1,3}P_{2,4}P_{5,7}P_{6,8}$,

$S_4 = P_{3,5}P_{4,6}P_{7,9}P_{8,10}$,

$S_5 = P_{9,1}P_{10,2}P_{3,6}P_{5,8}$,

$S_6 = P_{1,4}P_{2,5}P_{6,9}P_{7,10}$,

$S_7 = P_{4,7}P_{8,1}P_{9,2}P_{10,3}$.

Hence, W is computed as

$$\begin{pmatrix} 0.1094 & 0.1172 & 0.0781 & 0.0781 & 0.1250 & 0.1094 & 0.0781 & 0.1094 & 0.1172 & 0.0781 \\ 0.1094 & 0.1172 & 0.0781 & 0.0781 & 0.1250 & 0.1094 & 0.0781 & 0.1094 & 0.1172 & 0.0781 \\ 0.0859 & 0.1016 & 0.1094 & 0.1016 & 0.0938 & 0.1094 & 0.1016 & 0.0859 & 0.1016 & 0.1094 \\ 0.0859 & 0.1016 & 0.1094 & 0.1016 & 0.0938 & 0.1094 & 0.1016 & 0.0859 & 0.1016 & 0.1094 \\ 0.0938 & 0.0938 & 0.1172 & 0.1016 & 0.0938 & 0.0938 & 0.1016 & 0.0938 & 0.0938 & 0.1172 \\ 0.0938 & 0.0938 & 0.1172 & 0.1016 & 0.0938 & 0.0938 & 0.1016 & 0.0938 & 0.0938 & 0.1172 \\ 0.1016 & 0.0859 & 0.1094 & 0.1172 & 0.0781 & 0.0938 & 0.1172 & 0.1016 & 0.0859 & 0.1094 \\ 0.1016 & 0.0859 & 0.1094 & 0.1172 & 0.0781 & 0.0938 & 0.1172 & 0.1016 & 0.0859 & 0.1094 \\ 0.1094 & 0.1016 & 0.0859 & 0.1016 & 0.1094 & 0.0938 & 0.1016 & 0.1094 & 0.1016 & 0.0859 \\ 0.1094 & 0.1016 & 0.0859 & 0.1016 & 0.1094 & 0.0938 & 0.1016 & 0.1094 & 0.1016 & 0.0859 \end{pmatrix} \tag{6.3.9}$$

Nearest Neighbors(r=4)

The possible pairs of a 4-nearest neighbor network for n=10 can be expressed as

$$\begin{array}{l} (1,2)(2,3)(3,4)(4,5)(5,6)(6,7)(7,8)(8,9)(9,10)(10,1) \\ (1,3)(2,4)(3,5)(4,6)(5,7)(6,8)(7,9)(8,10)(9,1)(10,2) \\ (1,4)(2,5)(3,6)(4,7)(5,8)(6,9)(7,10)(8,1)(9,2)(10,3) \\ (1,5)(2,6)(3,7)(4,8)(5,9)(6,10)(7,1)(8,2)(9,3)(10,4) \end{array} \tag{6.3.10}$$

In this case, the chromatic index is either 8 or 9. Hence, optimal periodic sub-sequence (E) can be written as

$$E = E_1E_2E_3E_4E_5E_6E_7E_8,$$

where,

$$\begin{array}{l} E_1 = (1,2)(3,4)(5,6)(7,8)(9,10) \\ E_2 = (5,9)(3,6)(7,1)(10,4)(8,2) \\ E_3 = (9,2)(1,5)(6,10)(3,7)(4,8) \\ E_4 = (4,5)(1,3)(10,2)(6,7)(8,9) \\ E_5 = (2,4)(3,5)(6,8)(7,9)(1,10) \\ E_6 = (4,6)(5,7)(8,10)(9,1)(2,3) \\ E_7 = (1,4)(2,6)(7,10)(9,3)(5,8) \\ E_8 = (4,7)(2,5)(6,9)(8,1)(10,3) \end{array} \tag{6.3.11}$$

are disjoint sets.

Primitive gossip matrix (W) is expressed as

$W = S_1S_2S_3S_4S_5S_6S_7S_8S_9S_{10}$, where

$S_1 = P_{1,2}P_{3,4}P_{5,6}P_{7,8}P_{9,10}$,

$S_2 = P_{5,9}P_{3,6}P_{7,1}P_{10,4}P_{8,2}$,

$S_3 = P_{9,2}P_{1,5}P_{6,10}P_{3,7}P_{4,8}$,

$S_4 = P_{4,5}P_{1,3}P_{10,2}P_{6,7}P_{8,9}$,

$S_5 = P_{2,4}P_{3,5}P_{6,8}P_{7,9}P_{1,10}$,

$S_6 = P_{4,6}P_{5,7}P_{8,10}P_{9,1}P_{2,3}$,

$S_7 = P_{1,4}P_{2,6}P_{7,10}P_{9,3}P_{5,8}$,

$S_8 = P_{4,7}P_{2,5}P_{6,9}P_{8,1}P_{10,3}$.

Hence, W is computed as

$$\begin{pmatrix} 0.0977 & 0.1016 & 0.1016 & 0.0977 & 0.1016 & 0.1016 & 0.0977 & 0.0977 & 0.1016 & 0.1016 \\ 0.0977 & 0.1016 & 0.1016 & 0.0977 & 0.1016 & 0.1016 & 0.0977 & 0.0977 & 0.1016 & 0.1016 \\ 0.1055 & 0.0977 & 0.0977 & 0.1055 & 0.0977 & 0.0938 & 0.1055 & 0.1055 & 0.0938 & 0.0977 \\ 0.1055 & 0.0977 & 0.0977 & 0.1055 & 0.0977 & 0.0938 & 0.1055 & 0.1055 & 0.0938 & 0.0977 \\ 0.1016 & 0.0977 & 0.1016 & 0.1016 & 0.0977 & 0.0977 & 0.1016 & 0.1016 & 0.0977 & 0.1016 \\ 0.1016 & 0.0977 & 0.1016 & 0.1016 & 0.0977 & 0.0977 & 0.1016 & 0.1016 & 0.0977 & 0.1016 \\ 0.0977 & 0.1016 & 0.1016 & 0.0977 & 0.1016 & 0.1016 & 0.0977 & 0.0977 & 0.1016 & 0.1016 \\ 0.0977 & 0.1016 & 0.1016 & 0.0977 & 0.1016 & 0.1016 & 0.0977 & 0.0977 & 0.1016 & 0.1016 \\ 0.0977 & 0.1016 & 0.0977 & 0.0977 & 0.1016 & 0.1055 & 0.0977 & 0.0977 & 0.1055 & 0.0977 \\ 0.0977 & 0.1016 & 0.0977 & 0.0977 & 0.1016 & 0.1055 & 0.0977 & 0.0977 & 0.1055 & 0.0977 \end{pmatrix} \tag{6.3.12}$$

6.4 Proposed Method

To measure the convergence rate of the periodic gossip algorithms, it is essential to compute the second largest eigenvalue of the primitive gossip matrix of the network. However, calculating eigenvalues for large-scale networks is a computationally challenging task. To overcome this problem, the combination of *power-iteration* and *deflation* techniques [69] proposed in this chapter. By using the *power-iteration* method, we can obtain the largest eigenvalue and its corresponding eigenvector. To obtain the second largest eigenvalue, deflation technique has been used. In deflation technique, a rank one modification is applied to the original matrix to displace the largest eigenvalue. The power-iteration method can be applied to the deflated matrix to extract the second largest eigenvalue.

6.4.1 Power-Iteration Method

Let A be a $n \times n$ matrix, then to evaluate the largest eigenvalue follow the below steps.
(1)Choose a non-zero initial vector υ_0.
(2)For k=1,2....,evaluate $\upsilon_k = \frac{A\upsilon_{k-1}}{\alpha_k}$ where α_k is a component of the vector $A\upsilon_{k-1}$ which has the maximum modulus.

6.4.2 Deflation Technique

Let A be a $n \times n$ matrix with the eigenvalues $\lambda_1, \lambda_2, \lambda_3, ...\lambda_n$ and corresponding eigenvectors $\upsilon_1, \upsilon_2, \upsilon_3, ...\upsilon_n$.
(1)Evaluate the matrix

$$B = A - \lambda_1 \upsilon_1 x^T \tag{6.4.1}$$

where x is an arbitrary n-vector.
(2)Delete the first row of B and apply power-iteration method. The resultant eigenvalue and eigen vector are λ_2 and υ_2 respectively.

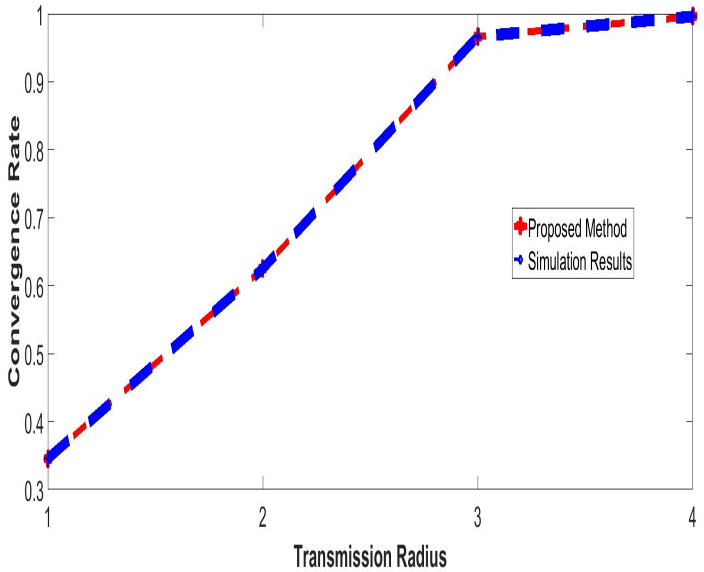

Figure 6.2: Effect of Node's Transmission Radius on Convergence rate for even number of nodes

By using the power-iteration and deflation techniques, convergence rates of the periodic gossip algorithm of the r-nearest neighbor ring network for n=9 and n=10 have been evaluated.

6.5 Numerical Results and Discussion

In this section, numerical results have been presented to investigate the impact of nodes' transmission radius on the convergence rate of average periodic gossip algorithms. Fig. 6.2 has been plotted for $n = 9$ varying r from 1 to 4. It has been observed that the convergence rate of periodic gossip algorithms increases with the node's transmission radius. At $r = 4$ convergence rate reaches approximately unity as the network will have full connectivity. Fig. 6.3 has been plotted for $n = 10$ varying the r from 1 to 4. As shown in the Fig. 6.3, convergence rate increases with the transmission radius and reaches unity for the odd number of nodes. We have verified the efficacy of the

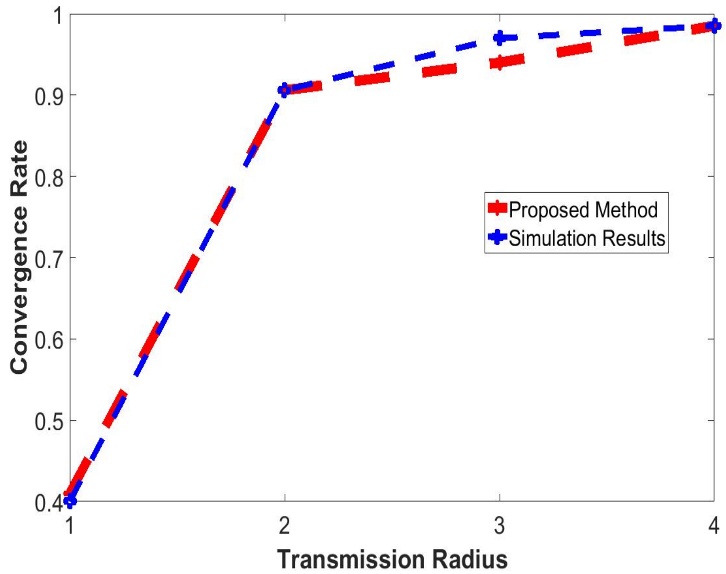

Figure 6.3: Effect of Node's Transmission Radius on Convergence rate for odd number of nodes

proposed method using MATLAB simulations.

6.6 Conclusions

In this chapter, WSN has been modeled as an r-nearest neighbor ring network, and the effect of transmission radius on the convergence rate of average periodic gossip algorithms have been investigated. Power-iteration and deflation techniques have been proposed to compute the convergence rate of average periodic gossip algorithms. Finally, numerical results illustrate that the convergence rate of periodic gossip algorithms increases with the nodes' transmission radius. The proposed method can also be used in computing the convergence rate of periodic gossip algorithms for large-scale WSNs.

Chapter 7

Conclusions and Future Work

This chapter discusses the major conclusions of the works presented in the thesis. Some of the studies in this thesis can be further explored. We discuss some of the open problems for further investigation.

7.1 Conclusions

This thesis studied the consensus algorithms for WSNs using regular graphs and one-dimensional lattice networks. In chapter 3, WSN has been modeled as r-nearest neighbor network and the closed-form expressions of convergence rate have been derived for average consensus algorithms. The variable *'r'* represents the nearest neighbors that can model the transmission radius or node overhead. From the numerical results shown, it is concluded that the convergence rate decreases with the number of nodes and network dimension. It has also been observed that the convergence rate increases with the nodes' transmission radius. Numerical results reveal the trade-off between power consumption and convergence rate. Finally, an optimization framework has been proposed to design the optimal transmission radius that maximizes the convergence rate and minimizes the power consumption.

Low power wireless networks such as WSNs consist of unreliable and time-varying channels. WSNs consists of asymmetric channels which cannot be modeled by the undirected graphs. In chapter 4, we study the effect of asymmetric links on the convergence rate of consensus algorithms for WSNs. WSN has been modeled as a directed graph, and the closed-form expressions of convergence rate have been derived for the ring, torus, r-nearest neighbor ring, and m-dimensional torus networks. Numerical results illustrated that the convergence rate decreases significantly with asymmetrical link factor in small-scale WSNs. In large-scale WSNs, the effect of asymmetrical links on convergence rate decreases with the number of nodes. Further, we studied the impact of the number of nodes, network dimension, and node overhead on the convergence rate of average consensus algorithms. This chapter also investigated the accuracy of directed graph modeling over undirected graph modeling using absolute error.

In chapter 5, WSN has been modeled as a one-dimensional lattice network, and the closed-form expressions of convergence rate have been derived for average periodic gossip algorithms. To investigate the effect of gossip weight on convergence rate, a generalized expression for convergence rate has been derived for periodic gossip algorithms. From the numerical results, it has been observed that w=0.9 gives faster convergence rate over w=0.5(average) for large-scale networks. Further, this chapter studied the effect of communication link failures on the convergence rate of periodic gossip algorithms. It has been observed that communication link failures drastically reduces the convergence rate of periodic gossip algorithms.

In chapter 6, WSN has been modeled as an r-nearest neighbor ring network, and the effect of the number of nodes and nearest neighbors on the convergence rate has been studied for average periodic gossip algorithms. To study the convergence rate for large-scale networks, this chapter proposes the power iteration and deflation techniques.

Theoretical results developed in this thesis can be utilized to measure the convergence time of the consensus algorithms for WSNs. Network diameter measures the number of steps required to obtain the global average in consensus algorithms. Hence, the convergence time of the two graphs is the equivalent if their diameter is the same. Therefore, theoretical results developed in this thesis can also be utilized to measure the convergence rate and convergence time of the arbitrary graph models.

7.2 Open Problems

(1) Increasing the nodes' transmission radius improves the convergence rate of consensus algorithms. However, the nodes' transmission radius is directly proportional to power consumption. Since, WSNs consist of limited power resources, obtaining the optimal radius to increase the convergence rate considering the node's power consumption is still an open problem.

(2) Deriving close-form expressions of convergence rate expression for a ring network, r-nearest neighbor ring network for periodic gossip algorithms are yet to be studied.

(3) To gain more insight in studying consensus algorithms for WSNs, the convergence rate can be further explored for other graph models such as random graphs, scale-free networks, and small world networks.

(4) This thesis mainly focused on studying deterministic consensus algorithms for WSNs. Randomized consensus algorithms are yet to be examined for WSN scenarios.

(5) In this thesis, we examined the effect of network size, network dimension, link failures, asymmetric links, and nearest neighbors on the convergence rate of consensus algorithms for wireless sensor networks. However, WSNs are spatial networks which

operate in hostile environments and unattended locations such as forests, hills, rivers, the human body, buildings, etc. So, it is necessary to study the WSN's dynamic behavior of physical phenomena over space. WSNs can be modeled by the spatial network models [70], [71]. As discussed in the [70], geometric graphs describe the simplest model of spatial networks which is obtained for a set of vertices located in the plane, and a set of edges constructed under the specific geometric condition. ErdosRenyi graph model is also a prominent spatial network model which describes the probability to connect any two nodes depends on the distance between them. In the WattsStrogatz model, the starting point is an n-dimensional lattice, and random links are added according to a given probability distribution [70]. Convergence rate of consensus algorithms can be studied for the aforementioned spatial models to investigate the effect of complex topological relationships over geometrical representations of physical features such as buildings, trees, and rivers in WSNs.

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